LIMITS, CONTINUITY AND DIFFERENTIABILITY

LIMITS

Left and Right Limit:

$\lim_{x\to a} f(x)$ exist if its left hand limit (LHL) at $x = a$, and right hand limit (RHL) at $x = a$ both are equal and finite.

i.e. $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \text{finite}$, otherwise limit does not exist.

Frequently used limits:

(i) If p(x) is a polynomial, $\lim_{x\to a} p(x) = p(a)$.

(ii) $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \cos x = 1$

(where 'x' is in radians)

(iii) $\lim_{x\to 0} (1+x)^{1/x} = e$

(iv) $\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^x = e$

(v) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(vi) $\lim_{x\to 0} \frac{a^x - 1}{x} = \ell n(a)$, $a \in R^+$.

(vii) $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

(viii) $\lim_{x\to a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$

(ix) $\lim_{x\to 0} \frac{\ell n(1+x)}{x} = 1$

(x) $\lim_{x\to 0} \frac{\log_a (1+x)}{x} = \log_a e$, $a > 0, \neq 1$

(xi) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0} \frac{\tan^{-1} x}{x}$

If $\lim_{x\to a} f(x) = 0$ then the following results would be holding true,

(i) $\lim_{x\to a} \frac{\sin f(x)}{f(x)} = \lim_{x\to a} \frac{\tan f(x)}{f(x)} = \lim_{x\to a} \cos f(x) = 1$

(ii) $\lim_{x\to a} \frac{\sin^{-1} f(x)}{f(x)} = \lim_{x\to a} \frac{\tan^{-1} f(x)}{f(x)} = 1$

(iii) $\lim_{x\to a} \frac{b^{f(x)} - 1}{f(x)} = \ln b \ (b > 0)$

(iv) $\lim_{x\to a} (1 + f(x))^{1/f(x)} = e.$

Frequently used Series Expansions:

Following are some of the frequently used series expansions:

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\dots\dots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\dots\dots$

$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots\dots\dots$

$\sin^{-1} x = x + \frac{1^2.}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \dots\dots\dots$

$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\dots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots$

$a^x = 1 + x.\ln a + (\ln a)^2 \frac{x^2}{2!} + \dots\dots\dots \qquad a \in R^+$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\dots,$ $n \in R.$ $|x|<1$, n is any real number

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots, \quad -1 < x \le 1.$

$(1+x)^{1/x} = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 \dots\dots\right)$

CONTINUITY

Definition of continuity:

A function $f(x)$ is said to be continuous at a point $x = a$ if and only if $f(a^-) = f(a^+) = f(a)$ = finite number, *i.e.*, $\lim\limits_{x \to a} f(x)$ exists finitely, $f(a)$ is a finite number and $= f(a)$.

More precisely, for given $\varepsilon > 0$ $\delta > 0$ such that $0 \le |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

Continuity of a Function in an Interval:

Continuity in an Open Interval

A function f(x) is said to be continuous in an open interval (a, b) if it is continuous at each point of (a, b).

Continuity in a Closed Interval

A function f(x) is said to be continuous in a closed interval [a, b] if

(i) f(x) is continuous from right at x = a i.e. $\lim\limits_{x \to a^+} f(x) = f(a)$.

(ii) f(x) is continuous from left at x = b i.e. $\lim\limits_{x \to b^-} f(x) = f(b)$ and;

(iii) f(x) is continuous at each point of the interval (a, b).

Properties of continuous functions:

Here we present two extremely useful properties of continuous functions;

Let $y = f(x)$ be a continuous function $\forall x \in [a, b]$, then following results hold true,

(i). f is bounded between a and b. That simply means that we can find real numbers m_1 and m_2 such $m_1 \le f(x) \le m_2 \forall x \in [a, b]$. It is customary to refer m_1 and m_2 as lower and upper bound of f (x) in [a, b]. Normally $m_1 \neq m_2$; in case $m_1 = m_2$, f (x) becomes a constant function in [a, b]

(ii). Every value between f (a) and f (b) will be assumed by the function atleast once. This property is also called intermediate value theorem of continuous function.
In particular if $f(a) \cdot f(b) < 0$, then f (x) will become zero atleast once in (a, b). It also means that if f (a) and f (b) have opposite signs then the equation $f(x) = 0$ will have atleast one real root in (a, b).

Continuity of Composite Functions:

If f is continuous at a and g is continuous at $f(a)$ then g *of* is continuous at a.

DIFFERENTIABILITY

Increment: Let y = f (x) be a given function. It is normal to accept that with a change in the value of independent variable, x in this case, there will also be a change in the corresponding values of the dependent variable y in this case. Say x_i and x_f be the initial and final values of independent variable.

Differential Coefficient:

Differential coefficient of y = f (x), with respect to x is defined as the limiting value of the ratio $\frac{\Delta y}{\Delta x}$ as Δx tends to zero. It is usually denoted by $\frac{dy}{dx}$ or f′ (x). Symbolically,

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

This formulae is also called the first principle of calculating the differential coefficient of any given function.

Differentiability of a function at a given point:

The function y = f (x) is said to be differentiable at a general point x, if and only if $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists finitely, else it is said to be non-differentiable at that point. It simply means that if y = f (x) has finite differential coefficient at a general point x i.e. f′ (x) exists finitely, f (x) is said to be differentiable, else non–differentiable.

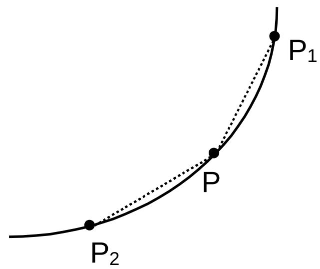
Clearly $\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ will exists finitely if $\lim\limits_{h\to 0^-}\frac{f(x-h)-f(x)}{-h}$ and $\lim\limits_{h\to 0^+}\frac{f(x+h)-f(x)}{h}$ are equal and finite.

Geometrical meaning of differentiability:

Let y = f (x) be a given function.

Let us take three points on the curve, namely

P (x, f (x)), P_1(x + h, f (x + h)) and P_2 (x –h, f (x –h)).



Now slope of the chord PP_1,

$$m_{PP_1}=\frac{f(x+h)-f(x)}{x+h-x}=\frac{f(x+h)-f(x)}{h}$$

Similarly slope of the chord PP_2,

$$m_{PP_2}=\frac{f(x)-f(x-h)}{x-(x-h)}=\frac{f(x-h)-f(x)}{-h}$$

Now as $h \to 0$, the points P_1 and P_2 would come close to P and chords PP_1 and PP_2 tend to become tangents that can be drawn to the curve at P.

In limiting case,

$m_{PP_1}=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ = f′ (x + 0) and $m_{PP_2}=\lim\limits_{h\to 0}\frac{f(x-h)-f(x)}{-h}$ = f′ (x – 0).

Differentiability in an interval:

(i). Differentiability in open interval

The function y = f (x) is said to be differentiable in (a, b), if it is differentiable at each point x ∈ (a, b).

(ii). Differentiability in closed interval

The function y = f (x) is said to be differentiable in [a, b] if f′ (a + 0), f′ (a –0) exist and f′(x) exists for all x ∈ (a, b).

- ***List of derivatives of important functions:***

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
4. $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\cos x) = -\sin x$
7. $\frac{d}{dx}(\tan x) = \sec^2 x$
8. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
9. $\frac{d}{dx}(\sec x) = \sec x \tan x$
10. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
15. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
16. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
18. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Rules for differentiation:

c is a constant and u, v are functions of x.

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(cu) = c\frac{du}{dx}$
3. $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$
4. $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$
5. $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$

6. $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u.\frac{dv}{dx}}{v^2}$

7. $\frac{d}{dx}(u^v)=u^v\left(\frac{v}{u}\frac{du}{dx}+\frac{dv}{dx}\ell nu\right)$

Differential Coefficient of function of function

If y = f(u) and u = g(x), then

$\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$ = f′(u) . g′(x) = f′(g(x)) . g′ (x)

Differentiation of implicit functions

Given the equation f(x,y) = c. For finding dy/dx, we differentiate both the sides w.r.t, x, treating y as a function of x, and then solve equation for dy/dx.

Differentiation of parametrically defined functions:

Working Rule

(i) If x and y are function of parameter t, then find dx/dt and dy/dt separately.

(ii) Then $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$.

e.g., x = a(θ+sinθ), y = a(1–cosθ) where θ is parameter.

- $\frac{dx}{d\theta}=a(1+\cos\theta)$, $\frac{dy}{d\theta}=a(0+\sin\theta)=a\sin\theta$
- $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\ \sin\theta}{a(1+\cos\theta)}=\frac{2\sin\frac{\theta}{2}\ \cos\frac{\theta}{2}}{2\ \cos^2\frac{\theta}{2}}=\tan\frac{\theta}{2}$

L'HOSPITAL'S RULE

Let f(x) and g(x) be two functions differentiable in the neighbourhood of the point a, except may be at the point 'a' itself.

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$.

Or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$.

Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ provided that the limit on the right exist as a finite number or is $\pm\infty$.

Use of Newton–Leibnitz's formula in evaluating the limits:

Let us consider the definite integral, $I(x) = \int_a^{g(x)} f(t)\,dt$, where 'a' is a given real number.

Newton Leibnitz's formula says that, $\frac{dI}{dx} = f(g(x)).\frac{d\,g(x)}{dx}$