INTEGRAL CALCULUS

Constant of Integration:

$\frac{d}{dx}(F(x)) = f(x) \Rightarrow \frac{d}{dx}[F(x)+c] = f(x)$.

Therefore, $\int f(x)\,dx = F(x) + c$.

Properties of Indefinite Integration:

(i) $\int af(x)dx = a\int f(x)dx.$

(ii) $\int (f(x)+g(x))\,dx = \int f(x)dx + \int g(x)dx.$

(iii) If $\int f(u)du = F(u) + c$, then $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+c,\ a \neq 0.$

Integration as the Inverse Process of Differentiation

Basic formulae:

Antiderivatives or integrals of some of the widely used functions (integrands) are given below.

- $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \Rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$
- $\frac{d}{dx}(\ln|x|) = \frac{1}{x} \Rightarrow \int \frac{1}{x}\,dx = \ln|x| + c$
- $\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x\,dx = e^x + c$
- $\frac{d}{dx}(a^x) = (a^x \ln a) \Rightarrow \int a^x\,dx = \frac{a^x}{\ln a} + c$ $(a > 0)$
- $\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x\,dx = \sin x + c$
- $\frac{d}{dx}(\cos x) = -\sin x \Rightarrow \int \sin x\,dx = -\cos x + c$
- $\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x\,dx = \tan x + c$

- $\frac{d}{dx}(\operatorname{cosec} x) = (-\cot x \operatorname{cosec} x) \Rightarrow \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$
- $\frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x \, dx = \sec x + c$
- $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x \Rightarrow \int \operatorname{cosec}^2 x \, dx = -\cot x + c$
- $\frac{d}{dx}(\sin^{-1}\frac{x}{a}) = \frac{1}{\sqrt{a^2 - x^2}} \Rightarrow \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\frac{d}{dx}(\tan^{-1}\frac{x}{a}) = \frac{a}{x^2 + a^2} \Rightarrow \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}} \Rightarrow \int \frac{1}{|x|\sqrt{x^2 - 1}} dx = \sec^{-1}(x) + c$
- $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + c$
- $\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x| + c$ or $\ln|\sec x| + c$
- $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \ln|\sec x + \tan x| + c$ or $\ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + c$
- $\int \operatorname{cosec} x \, dx = \int \frac{\operatorname{cosec} x(\cot x - \operatorname{cosec} x)}{\cot x - \operatorname{cosec} x} dx = \ln|(\cot x - \operatorname{cosec} x)| + c$ or $\ln\left|\tan\frac{x}{2}\right| + c$

Standard Formulae:

- $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left|x + \sqrt{x^2 + a^2}\right| + c$
- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + c$
- $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x - a}{x + a}\right| + c$
- $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + c$
- $\int \sqrt{u^2 + a^2} du = \frac{u}{2}\sqrt{u^2 + a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 + a^2}\right| + c$
- $\int \sqrt{u^2 - a^2} du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + c$
- $\int \sqrt{a^2 - x^2} \, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

Integration by Substitution:

There are following types of substitutions.

Direct Substitution:

If integral is of the form $\int f(g(x))\, g'(x)\, dx$, then put $g(x) = t$, provided $\int f(t)\, dt$ exists.

Standard Substitutions:

- For terms of the form $x^2 + a^2$ or $\sqrt{x^2 + a^2}$, put $x = a \tan\theta$ or $a \cot\theta$
- For terms of the form $x^2 - a^2$ or $\sqrt{x^2 - a^2}$, put $x = a \sec\theta$ or $a\, \text{cosec}\theta$
- For terms of the form $a^2 - x^2$ or $\sqrt{a^2 - x^2}$, put $x = a \sin\theta$ or $a \cos\theta$
- If both $\sqrt{a+x}$, $\sqrt{a-x}$ are present, then put $x = a \cos\theta$.
- For the type $\sqrt{(x-a)(b-x)}$, put $x = a\cos^2\theta + b\sin^2\theta$
- For the type $\left(\sqrt{x^2+a^2} \pm x\right)^n$ or $\left(x \pm \sqrt{x^2-a^2}\right)^n$, put the expression within the bracket = t.
- For the type $(x+a)^{-1-\frac{1}{n}}(x+b)^{-1+\frac{1}{n}}$ or $\left(\frac{x+b}{x+a}\right)^{\frac{1}{n}-1}\frac{1}{(x+a)^2}$ (n ∈N, n >1),put $\frac{x+b}{x+a} = t$.
- For $\frac{1}{(x+a)^{n_1}(x+b)^{n_2}}$, $n_1, n_2 \in N$ (and > 1), again put (x + a) = t (x + b)

Integration by Parts:

If u and v be two functions of x, then integral of product of these two functions is given by: $\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$

(Inverse, Logarithmic, Algebraic, Trigonometric, Exponential)

In the above stated order, the function on the left is always chosen as the first function. This rule is called as **ILATE** e.g. In the integration of $\int x\sin x dx$, x is taken as the first function and sinx is taken as the second function.

An important result: In the integral $\int g(x)e^x dx,$ if g(x) can be expressed as g(x) = f(x) + f′(x) then $\int e^x[f(x)+f'(x)]dx = e^x f(x) + c$

Integration By Partial Fractions:

A function of the form P(x)/Q(x), where P(x) and Q(x) are polynomials, is called a rational function. Consider the rational function $\frac{x+7}{(2x-3)(3x+4)} = \frac{1}{2x-3} - \frac{1}{3x+4}$

$Q(x) = (x - a)^{k...}(x^2 + \alpha x + \beta)^r$ ... where binomials are different, and then set

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + ... + \frac{A_k}{(x-a)^k} + \frac{M_1x+N_1}{x^2+\alpha x+\beta} + \frac{M_2x+N_2}{(x^2+\alpha x+\beta)^2} + ... + \frac{M_rx+N_r}{(x^2+\alpha x+\beta)^r} + ...$$

Algorithm to express the infinite series as definite integral:

(i) Express the given series in the form of $\sum \frac{1}{n} f\left(\frac{r}{n}\right)$

(ii)The limit when $n \to \infty$ is its sum $\lim_{h\to 0}\sum_{r=0}^{n-1}\frac{1}{n}.f\left(\frac{r}{n}\right)$

Replace *r/n* by *x*, 1/*n* by *dx* and $\lim\limits_{n\to\infty}\sum$ by the sign of integration

(iii) The lower and upper limits of integration will be the value of *r*/*n* for the first and last term (or the limits of these values respectively).

Some particular cases of the above are

(a) $\lim\limits_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$ or $\lim\limits_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}f\left(\frac{r}{n}\right)=\int_0^1 f(x)\,dx$,

(b) $\lim\limits_{n\to\infty}\sum_{r=1}^{pn}\frac{1}{n}f\left(\frac{r}{n}\right)=\int_\alpha^\beta f(x)\,dx$,

where $\alpha=\lim\limits_{n\to\infty}\frac{r}{n}=0$ (as r = 1) and $\beta=\lim\limits_{n\to\infty}\frac{r}{n}=p$ (as r = pn).

$\int_a^b f(x)dx = F(b)-F(a)$ (also called the Newton-Leibnitz formula).

The function F(x) is the integral of f(x) and a and b are the lower and the upper limits of integration.

Proof: From the first fundamental theorem

$\frac{d}{dx}\left[\int_a^x f(t)dt - F(x)\right] = f(x) - F'(x) = 0$ as $F'(x) = f(x)$ is given

i.e., the expression within the bracket must be constant in the interval and hence we can write

$F(x) = \int_a^x f(t)dt + c \;\; \forall x \in [a,b]$, where c is some real constant.

Thus, $F(b) = \int_a^b f(t)\,dt + c$ and $F(a) = \int_a^a f(t)dt + c = 0 + c = c$.

Hence, $\int_a^b f(t)dt = F(b) - F(a)$.

CHANGE OF VARIABLES IN DEFINITE INTEGRATION

If the functions f(x) is continuous on [a, b] and the function x = g(t) is continuously differentiable on the interval $[t_1, t_2]$ and a = g (t_1) and b = g (t_2), then

$$\int_a^b f(x)\,dx = \int_{t_1}^{t_2} f(g(t))\,g'(t)\,dt .$$

PROPERTIES OF DEFINITE INTEGRAL

1. Change of variable of integration is immaterial so long as limits of integration remain the same i.e. $\int_a^b f(x)dx = \int_a^b f(t)dt$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Where, the point c may lie between a and b or it may be exterior to (a, b).

4. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

5. $\int_0^a f(x)dx = \int_0^{a/2} [f(x)+f(a-x)]dx$

6. $\int_{-a}^a f(x)dx = \int_0^a [f(x)+f(-x)]dx$

7. $\int_a^b f(x)dx = (b-a)\int_0^1 f((b-a)x+a)dx$

8. If f (x) is a periodic function with period T, then

(a) $\int_a^{a+nT} f(x)dx = n\int_0^T f(x)dx, \text{ where } n \in I$

(b) $\int_{mT}^{nT} f(x)dx = (n-m)\int_0^T f(x)dx, \text{ where } n, m \in I$

(c) $\int_{a+nT}^{b+nT} f(x)dx = \int_a^b f(x)dx, \text{ where } n \in I$

Differentiation under the Integral Sign

Leibnitz's Rule:

If g is continuous on [a, b] and f_1 (x) and f_2 (x) are differentiable functions whose values lie in [a, b], then

$$\frac{d}{dx}\int_{f_1(x)}^{f_2(x)} g(t)dt = g(f_2(x))f_2(x) - g(f_1(x))f_1(x)$$