COMPLEX NUMBERS AND QUADRATIC EQUATIONS

A complex number, represented by an expression of the form $a + ib$ (a, b are real), If $z = a + ib$, then real part of z = Re $(z) = a$ and Imaginary part of $z = Im(z) = b$.

- ✓ If Re $(z) = 0$, the complex number is purely imaginary.
- ✓ If Im $(z) = 0$, the complex number is real.
- ✓

POLAR REPRESENTATION

Let $OP = r$, then $x = r\cos\theta$, and $y = r\sin\theta$

$\Rightarrow z = x + iy = r\cos\theta + ir\sin\theta$, $= r(\cos\theta + i\sin\theta)$. This is known as Trigonometric (or Polar) form of a complex Number. Here we should take the principal value of θ.

For general values of the argument

$z = r[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)]$ (where n is an integer)

PROPERTIES OF CONJUGATE

- $(\bar{z}) = z$
- $z = \bar{z} \Leftrightarrow z$ is real
- $z = -\bar{z} \Leftrightarrow z$ is purely imaginary
- $\text{Re}(z) = \text{Re}(\bar{z}) = \dfrac{z + \bar{z}}{2}$
- $\text{Im}(z) = \dfrac{z - \bar{z}}{2i}$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}} \quad (z_2 \neq 0)$

PROPERTIES OF MODULUS

- $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$.
- $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq |z|$.
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $z\bar{z} = |z|^2$
- $|z_1 z_2| = |z_1||z_2|$
  In general $|z_1 z_2 z_3 \ldots\ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots..|z_n|$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \qquad (z_2 \neq 0)$
- $|z_1 \pm z_2| \leq |z_1| + |z_2|$

ARGUMENT OF A COMPLEX NUMBERS

- $Z = 1 + i = (1, 1)$ and is marked by point $K(1, 1)$ lies in first quadrant.
  $\therefore |Z| = \sqrt{2}$ and arg $Z = \pi/4$.
- If $Z = 1 - i = (1, -1)$, then K lies in the fourth quadrant and $|Z| = \sqrt{2}$ and arg $Z = -\pi/4$.
- If $Z = -1 + i = (-1, 1)$, then K lies in the second quadrant and arg $Z = \dfrac{3\pi}{4}$.
- If $Z = -1 - i$, arg $Z = -\dfrac{3\pi}{4}$.

PROPERTIES OF ARGUMENTS

- $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi$ ($k = 0$ or 1 or -1)
  In general $\text{Arg}(z_1 z_2 z_3 \ldots.. z_n) = \text{Arg}(z_1) + \text{Arg}(z_2) + \text{Arg}(z_3) + \ldots\ldots + \text{Arg}(z_n) + 2k\pi$
  (where $k \in \boldsymbol{I}$)
- $\text{Arg}\left(\dfrac{z_1}{z_2}\right) = \text{Arg } z_1 - \text{Arg } z_2 + 2k\pi \qquad (k = 0$ or 1 or $-1)$
- $\text{Arg}\left(\dfrac{z}{\bar{z}}\right) = 2 \text{ Arg } z + 2k\pi \qquad (k = 0$ or 1 or $-1)$
- $\text{Arg}(z^n) = n \text{ Arg } z + 2k\pi \qquad (k = 0$ or 1 or $-1)$
- If $\text{Arg}\left(\dfrac{z_2}{z_1}\right) = \theta$, then $\text{Arg}\left(\dfrac{z_1}{z_2}\right) = 2k\pi - \theta$ where $k \in \boldsymbol{I}$.
- $\text{Arg } \bar{z} = -\text{Arg } z$
- If $\arg(z) = 0 \Rightarrow z$ is real.

De MOIVRE'S THEOREM

For any rational number *n*, the value or one of the values of $(\cos\theta + \sin\theta)^n$ is $(\cos n\theta + \sin n\theta)$. The following may also be noted:

(a) $(\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - \sin n\theta) = (\cos\theta - i\sin\theta)^n$

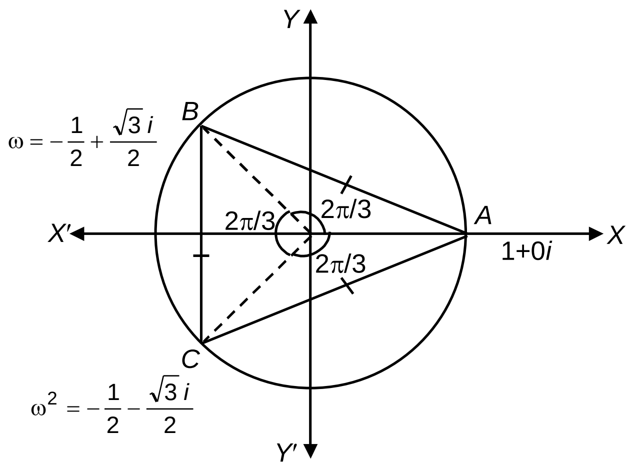
(b) $(\cos\theta + i\sin\theta)^{n} = (\cos n\theta + \sin n\theta) = (\cos\theta - i\sin\theta)^{-n}$

CUBE ROOTS OF UNITY

Consider the cubic (3rd degree) equation

$$x^3 = 1 = \cos 0 + i\sin 0 = \cos 2k\pi + i\sin 2k\pi$$

$$\therefore x = \sqrt[3]{1} = (\cos 2k\pi + i\sin 2k\pi)^{1/3} = \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$$



QUADRATIC EQUATIONS

An equation of the form $ax^2 + bx + c = 0$ ($a \neq 0$), *a*, *b*, *c* are real numbers, is called a quadratic equation.

The quantity $D = b^2 - 4ac$ is called the discriminant of quadratic equation

$$ax^2 + bx + c = 0 \quad (a \neq 0). \qquad \ldots(1)$$

The roots of the quadratic equation, generally denoted by α and β are

$$\alpha = \frac{-b + \sqrt{D}}{2a} \text{ and } \beta = \frac{-b - \sqrt{D}}{2a}.$$

NATURE OF THE ROOTS

1. Suppose $a, b, c \in R$ and $a \neq 0$. Then the following hold good:
   (a) The equation (1) has real and distinct roots if and only if $D > 0$.
   (b) The equation (1) has real and equal roots if and only if $D = 0$.
   (c) The equation (1) has complex roots with non-zero imaginary parts if and only if $D < 0$.

RELATION BETWEEN ROOTS AND COEFFICIENTS AND SYMMETRIC FUNCTIONS OF ROOTS

Let α and β be the roots of the equation $ax^2 + bx + c = 0$ then $\alpha + \beta = -\frac{b}{a}$ and

$\alpha\beta = \frac{c}{a}$

(a) if both roots are positive, then $\alpha + \beta = \frac{-b}{a} > 0$ and $\alpha\beta = \frac{c}{a} > 0$

(b) if both roots are negative, then $\alpha + \beta = \frac{-b}{a} < 0$ and $\alpha\beta = \frac{c}{a} > 0$

GRAPH OF QUADRATIC EXPRESSION

Let $f(x) = ax^2 + bx + c$ $\quad (a \neq 0, a, b, c \in \boldsymbol{R})$

It can be written as

$$y = f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right]$$

$$\Rightarrow \quad \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$ where, D is the discriminant.

This equation is of the form

$(x-\alpha)^2 = 4k\,(y-\beta)$ which represents a parabola with vertex at (α, β)

i.e., $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ in this case.

If $a > 0$, the parabola is concave upwards and if $a < 0$ the parabola is concave downwards.

QUADRATIC INEQUATIONS

The inequation of type $ax^2 + bx + c \geq 0$ or $ax^2 + bx + c \leq 0$ etc, $(a \neq 0)$ $a, b, c \in \boldsymbol{R}$ are known as quadratic inequations.

SOME RESULTS ON ROOTS OF A POLYNOMIAL EQUATION

(i) Factor theorem: If α is a root of the equation $f(x) = 0$, then $f(x)$ is exactly divisible by $(x-\alpha)$ and conversely, if $f(x)$ is exactly divisible by $(x-\alpha)$ then α is a root of the equation $f(x) = 0$ and the remainder obtained is $f(\alpha)$, which is zero.

(ii) Every equation of an odd degree has at least one real root.

(iii) If $x = \alpha$ is root repeated m times in $f(x) = 0$, ($f(x) = 0$ is an nth degree equation in x)
then $f(x) = (x-\alpha)^m g(x)$, where $g(x)$ is of degree $(n-m)$.

Rolle's Theorem:

This theorem is applicable to polynomials. It says that if $f(x)$ is a polynomial in the interval $[a, b]$ and $f(a) = f(b)$, then there is at least one point between a and b where $f'(x) = 0$.