MATRICES AND DETERMINANTS

MATRIX

A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of order m by n, written as m × n matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

TYPES OF MATRICES

Zero Matrix or Null Matrix

A matrix each of whose elements is zero, is called a zero matrix or a null matrix.

Square Matrix

A matrix in which number of rows is equal to the number of columns, say n, is called a square matrix of order n.

Diagonal Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is called a diagonal matrix if all the elements except those in the leading diagonal are zero, i.e., $a_{ij} = 0$ for $i \neq j$. In other words

$$A = \text{diag.}[a_{11} \quad a_{22} \quad a_{33} \quad \cdots \quad a_{nn}]$$

Unit Matrix

A square matrix in which every non-diagonal element is zero and every diagonal element is 1, is called a unit matrix or an identity matrix. Thus, a square matrix $A = [a_{ij}]_{n \times n}$ is a unit matrix if

$$a_{ij} = \begin{cases} 0 \text{ when } i \neq j \\ 1 \text{ when } i = j \end{cases}$$

ALGEBRA OF MATRICES

Addition of Matrices

Let A and B be two matrices each of order m × n. Then the sum matrix A + B is defined only if matrices A and B are of same order. The new matrix, say C = A + B is of order m × n and is obtained by adding the corresponding elements of A and B.

Subtraction of Matrices

Let A and B be two matrices of the same order. Then by A – B, we mean A + (–B). In other words, to find A – B we subtract each element of B from the corresponding element of A.

Multiplication of Matrices

Two matrices A and B can be multiplied only if the number of columns in A is same as the number of rows in B

TRANSPOSE OF A MATRIX

Let A be an m × n matrix. Then, the n × m matrix obtained by interchanging the rows and columns of A is called the transpose of A, and is denoted by A′ or A^1. Thus,

(i) if order of A is m × n, then, the order of A′ is n × m.

(ii) (i, j)the element of a = (j, i)the element of A′.

For example, if $A=\begin{bmatrix}2 & -3 & -1\\ 4 & 2 & 3\end{bmatrix}$, then $A'=\begin{bmatrix}2 & 4\\ -3 & 2\\ -1 & 3\end{bmatrix}_{3\times 2}$

SYMMETRIC MATRIX

A square matrix A is said to be symmetric if $A'=A$. That is, the matrix $A=\left[a_{ij}\right]_{n\times n}$ is said to be symmetric provided $a_{ij} = a_{ji}$ for all i and j.

SKEW SYMMETRIC MATRIX

A square matrix A is said to be skew symmetric, if $A' = -A$. That is, the matrix $A=\left[a_{ij}\right]_{n\times n}$ is skew-symmetric if $a_{ij}=-a_{ji}$ for all i and j.

ORTHOGONAL MATRIX

A square matrix of order n × n is said to be orthogonal if

$AA'=I_n=A'A.$

MINOR

If m – p rows and n – p columns from matrix $A_{m\times n}$, are removed, the remaining square submatrix of p rows and p columns is left. The determinant of a square submatrix of order p × p is called a minor of A of order p.

(i) every element of the matrix is the minor of order.

(ii) $\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix},\begin{vmatrix}3 & 6\\ 2 & 0\end{vmatrix},\begin{vmatrix}2 & 3\\ 4 & 1\end{vmatrix},\begin{vmatrix}0 & 4\\ 1 & 0\end{vmatrix}$ etc. are minors of order 2.

(iii) $\begin{vmatrix}3 & 1 & 0\\ 1 & 3 & 6\\ 1 & 2 & 0\end{vmatrix},\begin{vmatrix}2 & 1 & 0\\ 4 & 3 & 6\\ 8 & 2 & 0\end{vmatrix},\begin{vmatrix}2 & 3 & 1\\ 4 & 1 & 3\\ 8 & 1 & 2\end{vmatrix}$ etc. are the minors of order 3.

RANK OF A MATRIX

A positive integer r is said to be the rank of a non zero that A, if

(i) there exists atleast one minor in A of order which is zero,

(ii) every minor in A of order greater than r is zero, k is written as $\rho(A) = r$.

The rank of a zero matrix is defined to be zero.

Properties of Rank of a Matrix

(i) Rank of a matrix remains unaltered by elementary transformations.

(ii) No skew-symmetric matrix can be of rank 1.

(iii) Rank of matrix A = Rank of matrix A′.

SOLUTION OF A SYSTEM OF LINEAR EQUATIONS BY MATRIX METHOD

Consider a system of linear equations

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\
&\vdots\vdots\vdots \\
&a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n
\end{aligned}$$

We can express these equations as a single matrix equation

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{B}$$

Let $|A| \neq 0,$ so that A^{-1} exists uniquely. Pre-multiplying both sides of AX = B by A^{-1}, we get

$A^{-1}(AX) = A^{-1}B$ or $(A^{-1}A)X = A^{-1}B$

or $IX = A^{-1}B$ or $X = A^{-1}B$.

Hence $X = A^{-1}B$ is the unique solution of AX = B, |A| ≠ 0.

DETERMINANTs

MINOS AND COFACTORS

Minor of an Element of a Determinant

If we take an element of the determinant and delete the row and the column containing that element, the determinant left is called the minor of the element. It is denoted by M_{ij}

For example, given the 3 × 3 determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Then the minor of a_{11} of $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$;

the minor of a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and so on.

Cofactor of an Element of a Determinant

The cofactor C_{ij} of an element a_{ij} (the element in the *i*th row and *j*th column) is defined as $C_{ij} = (-1)^{i+j}M_{ij}$. It is denoted by C_{ij}. Thus,

$$C_{ij} = \begin{cases} M_{ij}, & \text{when } i + j \text{ is even} \\ -M_{ij}, & \text{when } i + j \text{ is odd} \end{cases}$$

For example, the cofactor of a_{12} in the 3 × 3 determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is $C_{12}=(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$.

PROPERTIES OF DETERMINANTS

1. The value of the determinant remains unchanged if rows are changed into columns and columns are changed into rows,
2. If two adjacent rows (columns) of a determinant are interchanged, the value of the determinant so obtained is the negative of the value of the original determinant, i.e.
3. If two rows or columns of a determinant are identical then its value is zero, i.e.,
4. If each element of a row or column of a determinant is multiplied by a constant k then the value of the new determinant is k times the value of the original determinant, i.e.
5. If any two rows or columns of a determinant are proportional, then its value is zero, i.e.

SOLUTION OF LINEAR EQUATIONS BY DETERMINANTS

1. **Cramer's Rule: Solution of system of linear equations in two unknowns**

   The solution of the system of equations

   $a_1x+b_1y=c_1$

   $a_2x+b_2y=c_2$

   is given by $x=\frac{D_1}{D}$ and $y=\frac{D_2}{D}$, where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

and $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$, provided $D \neq 0$.

2. Cramer's Rule: Solution of system linear equations in three unknowns

The solution of the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

is given by

$x = \frac{D_1}{D}, y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$, where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}; D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}; D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

provided $D \neq 0$.

(i) If $D \neq 0$, the system has only trivial solution ($x = y = z = 0$), and

(ii) If $D = 0$, the system has infinitely many solutions.