DIFFERENTIAL EQUATIONS

A differential equation is a mathematical equation that relates some function with its derivatives

Order and Degree of a Differential Equation:

The order of the highest differential coefficient appearing in the differential equation is called the order of the differential equation, while the exponent of the highest differential coefficient, when the differential equation is a polynomial in all the differential coefficients, is known as the degree of the differential equation.

Formation of Differential Equations:

Consider a family of curves f(x, y, α_1, α_2,…, α_n) = 0, where α_1, α_2,…,α_n are n independent parameters.

Solution of Differential Equation

First Order Differential Equations with Separable Variables

Let $\frac{dy}{dx}=\frac{M(x)}{N(y)} \Rightarrow$ N(y) dy = M(x) dx

Integrating both sides of (2), we get the solution *viz.*

$$\int N(y)\,dy = \int M(x)\,dx + c$$

Differential Equations Reducible to the Separable Variable Type:

A differential equation of the form $\frac{dy}{dx}=$ f (ax + by + c) is solved by writing ax + by + c = t.

Homogeneous Differential Equations:

Suppose we have a differential equation of the form $\frac{dy}{dx} = f(x,y)$, where f(x,y) is a function of x and y and is of the form $F\left(\frac{y}{x}\right)$ or $F\left(\frac{x}{y}\right)$.

These equations are solved by putting y = vx, where v ≡v (x), a function of x.

Equations Reducible to the Homogenous Form:

Equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{Ax+By+C}$

(aB≠ Ab) can be reduced to a homogenous form by changing the variables x, y to X, Y by writing x = X + h and y = Y + k; where h, k are constants to be chosen so as to make the given equation homogenous. We have

$\frac{dy}{dx} = \frac{d(Y+k)}{d(X+h)} = \frac{dY}{dX}$. Hence the given equation becomes $\frac{dY}{dX} = \frac{aX + bY + (ah+bk+c)}{AX+BY+(Ah+Bk+C)}$

Let h and k be so chosen as to satisfy the relation ah + bk + c = 0 and Ah + Bk + C =0.

These give $h = \frac{bC-Bc}{aB-Ab}$ $k = \frac{Ac-aC}{aB-Ab}$

which are meaningful except when aB = Ab.

$\frac{dY}{dX} = \frac{aX+bY}{AX+BY}$ can now be solved by means of the substitution Y = VX

In case aB = Ab, we write ax + by = t.

First Order Linear Differential Equations:

The most general form of this category of differential equations is $\frac{dy}{dx} + Py = Q$, , where P and Q are functions of x alone.

We use here an Integrating factor, namely $e^{\int p dx}$ and the solution is obtained by

$y\, e^{\int p dx} = \int Q\, e^{\int p dx} dx + c$, where c is an arbitrary constant.

General Form of Variable Separation

(i) $d(x + y) = dx + dy$

(ii) $d(xy) = y\,dx + x\,dy$

(iii) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

(iv) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

(v) $d(\log xy) = \frac{ydx + xdy}{xy}$

(vi) $d\left(\log \frac{y}{x}\right) = \frac{(xdy - ydx)}{xy}$

(vii) $d\left(\frac{1}{2}\log \frac{x+y}{x-y}\right) = \frac{x\,dy - y\,dx}{x^2 - y^2}$

(viii) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$

(ix) $\frac{d\left[f(x,y)\right]^{1-n}}{1-n} = \frac{f'(x,y)}{\left(f(x,y)\right)^n}$

(x) $d\left(\sqrt{x^2 + y^2}\right) = \frac{xdx + ydx}{\sqrt{x^2 + y^2}}$

ORTHOGONAL TRAJECTORY

Any curve, which cuts every member of a given family of curves at right angle, is called an orthogonal trajectory of the family. For example, each straight line passing through the origin, i.e. $y = kx$ is an orthogonal trajectory of the family of the circles $x^2 + y^2 = a^2$

APPLICATION OF DIFFERENTIAL EQUATIONS

Geometrical Applications

We also use differential equations for finding the family of curves for which some conditions involving the derivatives are given. For this we proceed in the following way:

Equation of the tangent at a point (x, y) to the curve y = f(x) is given by

$Y - y = \frac{dy}{dx}(X - x)$, At the X axis, Y = 0, and $X = x - y/\frac{dy}{dx}$.

At the Y axis, X = 0, and $Y = y - x\frac{dy}{dx}$.

Similar information can be obtained for normals by writing its equation as $(y - y)\frac{dy}{dx} + (X - x) = 0$.