# Co-ordinate Geometry

## DISTANCE FORMULA

The distance between two points $P$ and $Q$ having coordinates $(x_1, y_1)$ and $(x_2, y_2,)$ is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

## SECTION FORMULA

The coordinates of the point $R$ which divides the line joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $m : n$ are given by

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

The division may be internal or external

## 1. Area of polygon

(a) If the vertices of a triangle are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ then area = |A|, where

$$A = \frac{1}{2}\left(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right)$$

2. Different centres of a triangle ABC having vertices

$A \equiv (x_1, y_1), B \equiv (x_2, y_2), C \equiv (x_3, y_3)$

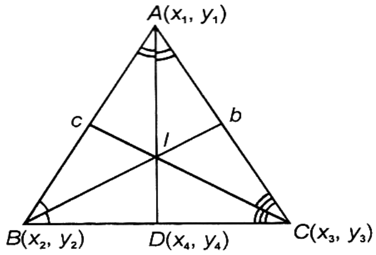
(a) **Centroid (G):**

Centroid of a triangle is the point of intersection of medians.

$$G \equiv \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

(b) **Incentre (I):**

Incentre is the point of intersection of bisectors of internal angles.




$$I \equiv \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

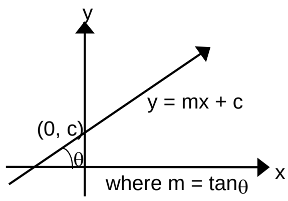
## 3. Form Straight lines

- **Slope-Intercept Form :**

  y = mx + c,

  where m = slope of the line = tanθ

  c = y intercept

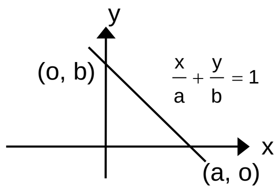



- **Intercept Form :**

  x/a + y/b = 1

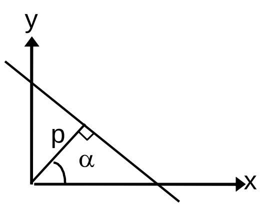
  x intercept = a

  y intercept = b




- **Normal Form :**

  x cosα + y sinα = p, where α is the angle which the perpendicular to the line makes with the axis of x and p is the length of the perpendicular from the origin to the line. p is always positive.



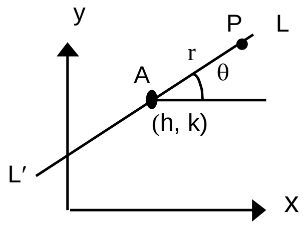
- **Slope Point Form:**

  Equation: $y - y_1 = m(x - x_1)$, is the equation of line passing through the point $(x_1, y_1)$ and having the slope 'm'

- **Two Points Form:**

  Equation: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, is the equation of line passing through two points
  $(x_1, y_1)$ and $(x_2, y_2)$.

- **Parametric Form:**




To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis. P(x, y) is any point on the line LAL′.

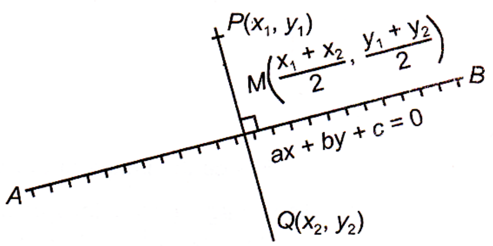
Let AP = r. $x - h = r\cos\theta$, $y - k = r\sin\theta$

$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$ is the equation of the straight line LAL′.

Any point on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k).

**Image Reflection, Foot of perpendicular, perpendicular distance of a point with respect to a line.**

(i) Let $Q(x_2, y_2)$ be the image of $P(x_1, y_1)$ with respect to the line mirror $ax + by + c = 0$. Then (slope of PQ) × (slope of AB) = –1




$$\Rightarrow \frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$$

(ii) The foot of the perpendicular from $(x_1, y_1)$ to the line $ax + by + c = 0$ is given by $\Rightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-(ax_1 + by_1 + c)}{a^2 + b^2}$

(iii) The length of the perpendicular from $(x_1, y_1)$ to the line $ax + by + c = 0$ is given by $p = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

## EQUATIONS OF THE CIRCLE IN VARIOUS FORMS:

- The simplest equation of the circle is $x^2 + y^2 = r^2$ whose centre is (0, 0) and radius r.
- The equation $(x - a)^2 + (y - b)^2 = r^2$ represents a circle with centre (a, b) and radius r.
- The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ is the general equation of a circle with centre (-g, -f) and radius $\sqrt{g^2 + f^2 - c}$.
- Equation of the circle with points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as extremities of a diameter is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

- The equation of the circle through three non-collinear points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ is $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$.

**Parametric Equation of a Circle**: The equations $x = a\cos\theta$, $y = a\sin\theta$ are called parametric equations of the circle $x^2 + y^2 = a^2$ and $\theta$ is called a parameter. The point $(a\cos\theta, a\sin\theta)$ is also referred to as point $\theta$. The parametric coordinates of any point on the circle $(x - h)^2 + (y - k)^2 = a^2$ are given by $(h + a\cos\theta, k + a\sin\theta)$ with $0 \le \theta < 2\pi$.

**The Position of a Point with respect to a Circle :** The point $P(x_1, y_1)$ lies outside, on, or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >$ = or < 0.

**Equations of Tangents and Normals:** If $S = 0$ be a curve then $S_1 = 0$ indicate the equation which is obtained by substituting $x = x_1$ and $y = y_1$ in the equation of the given curve and $T = 0$ is the equation whichis obtained by substituting $x^2 = xx_1$, $y^2 = yy_1$, $2xy = xy_1 + yx_1$, $2x = x + x_1$, $2y = y + y_1$ in the equation $S = 0$.
If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ then $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$, and
$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

- Equation of the tangent to the circle
  $x^2 + y^2 + 2gx + 2fy + c = 0$ at $A(x_1, y_1)$ is
  $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.
- The condition that the straight line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$ is
  $c^2 = a^2(1 + m^2)$ and the point of contact is $(-a^2m/c, a^2/c)$ i.e.
  $y = mx \pm a\sqrt{1+m^2}$ is always a tangent to the circle $x^2 + y^2 = a^2$ whatever be the value of m.
- The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle
  $x^2 + y^2 + 2gx + 2fy + c = 0$ is $T^2 = SS_1$ .
- The equation of the normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point $(x_1, y_1)$ lying on the circle is $\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$.
- The equation of the chord of the circle $S \equiv 0$, with mid-point $(x_1, y_1)$ is $T = S_1$.
- The length of the tangent drawn from a point $(x_1, y_1)$ outside the circle $S \equiv 0$, is $\sqrt{S_1}$.
- Locus of point of intersection of two mutually perpendicular tangents to the circle is called director circle. For circle $x^2 + y^2 = a^2$, equation of director circle is given by $x^2 + y^2 = 2a^2$.

**Chord of contact:** From a point $P(x_1, y_1)$ two tangents PA and PB can be drawn to the circle. The chord AB joining the points of contact A and B of the tangents from P is called the chord of contact of $P(x_1, y_1)$ with respect to the circle. Its equation is given by $T = 0$.

**Radical Axis:** The radical axis of two circles is the locus of a point from which the tangent segments to the two circles are of equal length.

**Equation to the Radical Axis:** In general S - S' = 0 represents the equation of the Radical Axis to the two circles . i.e. $\mathbf{2x(g - g') + 2y(f - f') + c - c' = 0}$ where $S \equiv x^2 + y^2 + 2gx + 2fy + c=0$ and $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' =0$

**Family of circles**

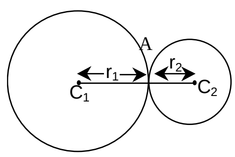
**Equation of a Circle through Intersection of Two Circles**

(i) If $S_1 \equiv 0$ and $S_2 \equiv 0$ be two circle, intersecting in real points, then $S_1 + \lambda S_2 = 0$ $(\lambda \neq -1)$ is the equation of the family of circles passing through the common points of $S_1 \equiv 0$ and $S_2 \equiv 0$.

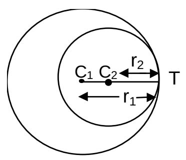
(ii) If $S_1 \equiv 0$ and $S_2 \equiv 0$ intersect, then $S_1 - S_2 \equiv 0$ is the equation of their common chord.

**External and Internal Contacts of Circles**

There are $r_1$ and $r_2$ two radii of two circles with centers $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$ then $C_1C_2 = r_1 + r_2$. Coordinates of the point of contact are




$$A \equiv \left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right).$$



## Common Tangents to Two Circles

Let circles $S_1 \equiv 0$ and $S_2 \equiv 0$ have centers $O_1$ and $O_2$ respectively and radii $r_1$ and $r_2$ respectively. The number of common tangents of $S_1 \equiv 0$ and $S_2 \equiv 0$ is

(i) **zero,** if one circle lies totally inside the other

(ii) **one,** if circles touch each other internally

(iii) **two,** if circles intersect at two distinct points

(iv) **three,** if circles touch each other externally

## CONIC SECTION

A conic section is the locus of a point such that the ratio of distances from fixed point to the fixed line is always constant. The fixed point is called the focus and the fixed line is called the directrix of the conic. The constant ratio is called the eccentricity of the conic and is denoted by e. If

- e = 1, the conic is called Parabola.
- e < 1, the conic is called Ellipse.
- e > 1, the conic is called Hyperbola.
- e = 0, the conic is called Circle.
- e = ∞, the conic is called pair of straight lines.

Important Terms

**1. Axis**

The straight line passing through the focus and perpendicular to the directrix of the conic is known as its axis.

**2. Vertex**

A point of intersection of a conic with its axis is known as a vertex of the conic.

**3. Focal Chord**

A chord passing through the focus is known as focal chord of the conic.

### 4. Latus Rectum

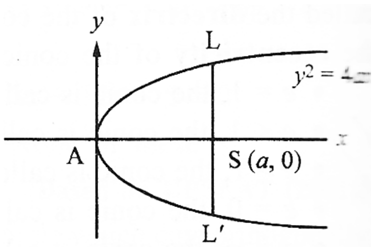
The focal chord which is perpendicular to the axis is known as latus rectum of the conic.

### Double Ordinate

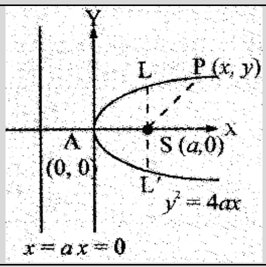
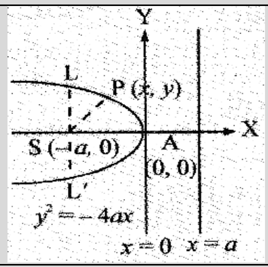
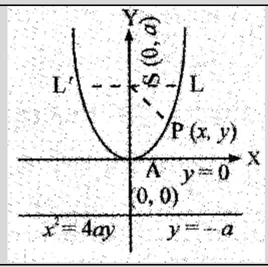
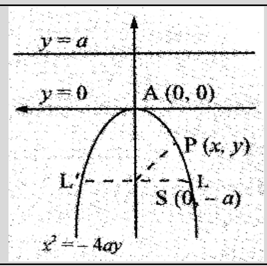
A chord of the conic which is perpendicular to the axis is called the double ordinate of the conic.

## PARABOLA

A parabola is the locus of a point such that the distances from a fixed point to the fixed line is always equal. Fixed point is called as focus and the fixed line is called as directrix.




Four Standard Forms of the Parabola

| Standard Equation | $y^2 = 4ax(a > 0)$ | $y^2 = -4ax(a > 0)$ | $x^2 = 4ay(a > 0)$ | $x^2 = -4ay(a > 0)$ |
|---|---|---|---|---|
| Slope of the Parabola |   |   |   |   |
| Vertex | A(0, 0) | A(0, 0) | A(0, 0) | A(0, 0) |
| Focus | S (a, 0) | S (–a, 0) | S (0, a) | S (0, – a) |
| Equation of directrix | x = – a | x = a | y = – a | y = a |
| Equation of axis | y = 0 | y = 0 | x = 0 | x = 0 |

| | | | | |
|---|---|---|---|---|
| Length of latus rectum | 4a | 4a | 4a | 4a |
| Extermities of latus rectum | (a, ± 2a) | (–a, ± 2a) | (± 2a, a) | (± 2a, –a) |
| Equation of latus rectum | x = a | x = –a | y = a | y =– a |
| Equation of tangents at vertex | x = 0 | x = 0 | y = 0 | y = 0 |
| Focal distance of a point P(x, y) | x + a | x – a | y + a | y – a |
| Parametric coordinates | $(at^2, 2at)$ | $(-at^2, 2at)$ | $(2at^2, at)$ | $(2at^2, -at)$ |
| Ecentricity (e) | 1 | 1 | 1 | 1 |

## Condition for Tangency and Point of Contact

The $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$ and the coordinates of the coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

## EQUATIONS OF TANGENT

### 1. Point Form

The equation of the tangent to the parabola $y^2 = 4ax$ at the point $(x_1, y_1)$ is

$$yy_1 = 2a(x + x_1)$$

## POINT OF INTERSECTION OF TANGENTS

The point of intersection of tangents drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

# EQUATIONS OF NORMAL

## 1. Point Form

The equation of the normal to the parabola $y^2 = 4ax$ at a point $(x_1, y_1)$ is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

## Condition for Normality

The line $y = mx + c$ is a normal to the parabola $y^2 = 4ax$ if $c = -2am - am^3$.

## Point of Intersection of Normals

The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

$$R \equiv \left[2a + a\left(t_1^2 + t_2^2 + t_1t_2\right), -at_1t_2(t_1 + t_2)\right].$$

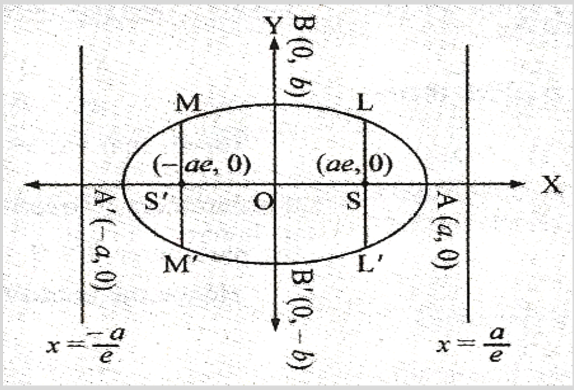
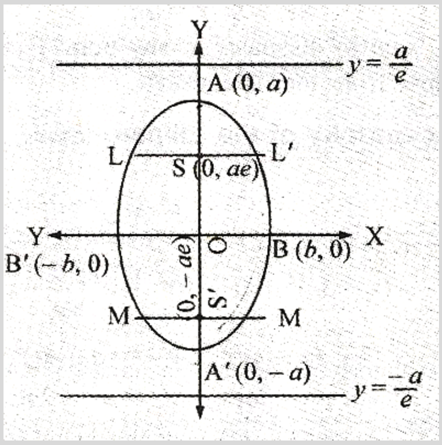
## Chord of Contact

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $T = 0$ where $T \equiv yy_1 - 2a(x + x_1)$.

## Equation of Diameter of a Parabola

The equation of the diameter bisecting chords of slope m of the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.

# TWO STANDARD FORMS OF THE ELLIPSE

| | | |
|---|---|---|
| Standard equation | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b)$ (Horizontal Form of an Ellipse) | $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1(a, b)$ (Vertical From of an ellipse) |
| Shape of the Ellipse |   |   |
| Centre | (0, 0) | (0, 0) |
| Equation of major axis | y = 0 | x = 0 |
| Equation of minor axis | x = 0 | y = 0 |
| Length of major axis | 2a | 2a |
| Length of minor axis | 2b | 2b |
| Foci | (±ae, 0) | (0, ±ae) |
| Vertices | (±a, 0) | (0, ±a) |
| Equation of directrices | $x=\pm\frac{a}{e}$ | $y=\pm\frac{a}{e}$ |
| Eccentricity | $e=\sqrt{\frac{a^2-b^2}{a^2}}$ | $e=\sqrt{\frac{a^2-b^2}{a^2}}$ |
| Length of latus rectum | $\frac{2b^2}{a}$ | $\frac{2b^2}{a}$ |
| Ends of latra-recta | $\left(\pm ae, \pm\frac{b^2}{a}\right)$ | $\left(\pm\frac{b^2}{a}, \pm ae\right)$ |
| Parametric coordinates | (a cos θ, b sin θ) | (a cos θ, b sin θ) |
| Focal radiis | P = a – $ex_1$ and S′P = a + $ex_1$ | SP = a – $ey_1$ and S′P = a + $ey_1$ |
| Sum of focal radii SP + S′P = | 2a | 2a |
| Distance between foci | 2ae | 2ae |
| Distance between directrices | $\frac{2a}{e}$ | $\frac{2a}{e}$ |
| Tangent at the vertices | x = ± a | y = ± a |

## EQUATION OF TANGENT

The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ at the point ($x_1$, $y_1$) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

## EQUATION OF NORMAL

The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ at the point ($x_1$, $y_1$) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2.$ **Condition for Normality**

The line y = mx + c is a normal to the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c^2=\frac{m^2\left(a^2-b^2\right)^2}{\left(a^2+b^2m^2\right)}$

## CHORD OF CONTACT

The equation of chord of contact of tangent drawn from a point P($x_1$, $y_1$) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is T = 0, where $T\equiv\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

## Equation of a Diameter

The equation of the diameter bisecting chords of slope m of the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=-\frac{b^2}{a^2m}x.$

# HYPERBOLA

| | Hyperbola | Conjugate Hyperbola |
|---|---|---|
| Standard equation | $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ | $\frac{-x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ |
| Centre | (0, 0) | (0, 0) |
| Equation of transverse axis | y = 0 | x = 0 |
| Equation of conjugate axis | x = 0 | y = 0 |
| Length of Transverse axis | 2a | 2b |
| Length of Conjugate axis | 2b | 2a |
| Foci | $(\pm ae, 0)$ | $(0, \pm be)$ |
| Equation of directrices | $x=\pm\frac{a}{e}$ | $y=\pm\frac{b}{e}$ |
| Vertices | $(\pm a, 0)$ | $(0, \pm b)$ |
| Eccentricity | $e=\sqrt{\frac{a^2+b^2}{a^2}}$ | $e=\sqrt{\frac{a^2+b^2}{a^2}}$ |
| Length of latus rectum | $\frac{2b^2}{a}$ | $\frac{2a^2}{b}$ |
| Parametric Coordinates | $(a \sec\theta, b \tan\theta)$ | $(b \sec\theta, a \tan\theta)$ |
| Focal radii | $SP = ex_1 - a$ and $S'P = ex_1 + a$ | $SP = ey_1 - b$ and $S'P = ey_1 + b$ |
| Difference of focal radii (S′P – SP) | 2a | 2b |
| Tangents at the vertices | $x = \pm a$ | $y = \pm b$ |

## Equation of Tangent

The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(x_1, y_1)$ is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

## Equations of Normal

The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $(x_1, y_1)$ is $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$.

## CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is T = 0, where $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1.$

## EQUATION OF A DIAMETER OF A HYPERBOLA

The equation of the diameter bisecting chords of slope m of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $y=\frac{b^2}{a^2m}x.$

## Rectangular Hyperbola

If asymptotes of the standard hyperbola are perpendicular to each other is called a rectangular hyperbola.
The equation of rectangular hyperbola $xy = c^2$, where c is any constant and eccentricity $e = \sqrt{2}$.