THREE DIMENSIONAL GEOMETRY

DISTANCE FORMULA

The distance between two points P and Q having coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

SECTION FORMULA

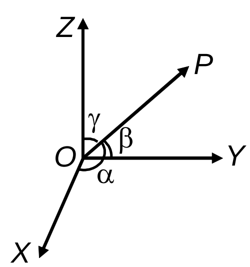
The coordinates of the point R which divides the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m : n$ are given by

$$\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}$$

The division may be internal or external

DIRECTION COSINES

If the position vector of a point P i.e., $\overrightarrow{OP}$ makes angles α, β and γ with the positive direction of x, y and z axis respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called its direction cosines. They are also denoted by l, m and n respectively.



i.e., $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

DIRECTION RATIOS

If a, b, c three numbers such that $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$

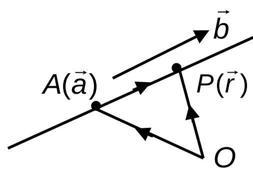
where l, m, n are direction cosines of a vector $\vec{r}$, then a, b, c are known as direction numbers or direction ratios of $\vec{r}$.

Two vectors having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 are parallel if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$.

They are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

VECTOR EQUATION OF A LINE PASSING THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

Let A be a fixed point having position vector $\vec{a}$ and the line is parallel to the vector $\vec{b}$. P is an arbitrary point having position vector $\vec{r}$ on the line.



From ΔOAP, $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}$

$\Rightarrow \quad \vec{r}=\vec{a}+\lambda\vec{b}$

This is the required equation of line. λ is an arbitrary real number.

CARTESIAN EQUATION OF A LINE PASSING THROUGH A GIVEN POINT AND GIVEN DIRECTION RATIOS

Let $A\ (x_1, y_2, z_3)$ be the fixed point and the line has direction ratios p_1, p_2, p_3.

$$\frac{x-x_1}{p_1}=\frac{y-y_2}{p_2}=\frac{z-z_3}{p_3}=\lambda$$

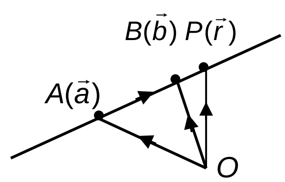
This is the Cartesian equation of the line also called symmetrical form of line.

Any point on this line can be taken as

$$(x_1 + p_1\lambda,\ x_2 + p_2\lambda,\ x_3 + p_3\lambda).$$

VECTOR EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

Let A and B be two fixed points having position vectors $\vec{a}$ and $\vec{b}$. P is a variable point on the line.



From ΔOPA again, $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}$

$\Rightarrow \quad \overrightarrow{OP}=\overrightarrow{OA}+\lambda(\overrightarrow{AB})$

$\Rightarrow \quad \vec{r}=\vec{a}+\lambda\,(\vec{b}-\vec{a})$

This is the required equation.

CARTESIAN EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

If coordinates of A and B are (x_1, y_1, z_1) and (x_2, y_2, z_2), the Cartesian equation is given by

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

ANGLE BETWEEN TWO LINES

If two lines are parallel to vectors $\vec{b}_1$ and $\vec{b}_2$, the angle between them is given by

$$\cos\theta=\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1|\,|\vec{b}_2|}$$

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SHORTEST DISTANCE BETWEEN TWO LINES

If $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ are two skew lines, the shortest distance between them is the perpendicular distance.

$$d=\left|\frac{(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|=\left|\frac{[\vec{b}_1\,\vec{b}_2\,(\vec{a}_2-\vec{a}_1)]}{|\vec{b}_1\times\vec{b}_2|}\right|$$

Clearly two lines intersect if $[\vec{b}_1\vec{b}_2\;(\vec{a}_2-\vec{a}_1)]=0$

If the lines are parallel,

i.e., $\vec{r}=\vec{a}_1+\lambda\vec{b}$

and $\vec{r}=\vec{a}_2+\lambda\vec{b}$

the formula to calculate shortest distance becomes

$$d=\frac{|(\vec{a}_2-\vec{a}_1)\times\vec{b}|}{|\vec{b}|}$$

PERPENDICULAR DISTANCE OF A POINT FROM A LINE

The perpendicular distance d can be obtained using vector form as well as Cartesian form of the line.

Let the line be $\vec{r} = \vec{a} + \lambda\vec{b}$ and p

be the point

$$\Rightarrow d = \left| \frac{(\vec{p} - \vec{a}) \times \vec{b}}{|\vec{b}|} \right|$$

EQUATION OF A PLANE PASSING THROUGH THREE GIVEN POINTS

If three non-collinear points are given, there is a unique plane passing through them. Let the points be *A*, *B* and *C* having position vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Then $\overrightarrow{AB}$ and $\overrightarrow{BC}$ lie in the plane. So, as in the previous article the equation of plane becomes

$$\vec{r} = \vec{a} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC} \quad \text{or} \quad \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) + \mu(\vec{c} - \vec{a})$$

$$\Rightarrow \quad \vec{r} = (1 - \lambda - \mu)\ \vec{a} + \lambda\vec{b} + \mu\vec{c}$$

INTERCEPT FORM OF A PLANE

This is a special case of the previous article. The equation of a plane intercepting the coordinate axes at (*a*, 0, 0), (0, *b*, 0) and (0, 0, *c*) is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

ANGLE BETWEEN TWO LINES

The angle between two planes is defined as the angle between their normals. If $\vec{n}_1$ and $\vec{n}_2$ are the normals and θ is the angle then

$$\cos\theta = \frac{\vec{n}_1.\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Obviously, two planes are perpendicular if $\vec{n}_1 . \vec{n}_2 = 0$.

They are parallel if $\vec{n}_1 = \lambda\vec{n}_2$ where λ is a scalar.

FAMILY OF PLANES

PLANE PARALLEL TO A GIVEN PLANE

Since parallel planes have the same normal vector, so equation of a plane parallel to $\vec{r}.\hat{n} = d_1$ is of the form $\vec{r}.\hat{n} = d_2$, where d_2 is determined by the given conditions.

In Cartesian form, if $ax + by + cz + d = 0$ be the given plane then the plane parallel to this plane is $ax + by + cz + k = 0$.

PLANE PASSING THROUGH INTERSECTION OF TWO PLANES

Two planes intersect in a line if they are not parallel. Any plane through the line of intersection of two planes can be written as

$$(\vec{r}.\vec{n}_1 - d_1) + \lambda(\vec{r}.\vec{n}_2 - d_2) = 0$$

DISTANCE OF A POINT FROM A PLANE

The perpendicular distance of a point $P(\vec{p})$ from the plane $\vec{r}.\vec{n} = d$ is given by $\frac{|\vec{p}.\vec{n} - d|}{|\vec{n}|}$

In Cartesian form, the perpendicular distance of $P(x_1, y_1, z_1)$ from the plane

$ax + by + cz + d = 0$ is equal to $\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

DISTANCE BETWEEN PARALLEL PLANES

The distance between $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

BISECTORS OF TWO PLANES

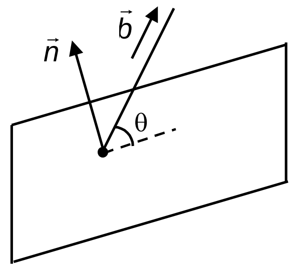
If the equations of planes are $a_1x + b_1y + c_1z + d_1 = 0$ and $a_1x + b_1y + c_1z + d_2 = 0$

the planes bisecting these angles are given by

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

ANGLE BETWEEN A LINE AND A PLANE

Let the line be $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane be $\vec{r}.\vec{n} = d$. If θ is the angle between them then



$$\cos(90° - \theta) = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}$$

$$\Rightarrow \quad \sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}$$

In Cartesian form, if the plane is $ax + by + cz + d = 0$ and line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

then $\sin\theta = \dfrac{al + bm + cn}{\sqrt{a^2 + b^2 + c^2}\sqrt{l^2 + m^2 + n^2}}$

so, the condition that line is parallel to the plane is $\vec{b}.\vec{n} = 0$ or $al + bm + cn = 0$

and the condition of perpendicularity is $\vec{b} = \lambda\vec{n}$ or $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

CONDITION FOR A LINE TO LIE IN A PLANE

Vector form: $\vec{a}.\vec{n} = d$ and $\vec{b}.\vec{n} = 0$