Exercise Set 12.1 (Solutions)

Question 1

1. Let ABCD be a quadrilateral.
   (i) List all sides of ABCD adjacent to side AB. List all sides opposite to AB.

   (ii) List all angles of ABCD adjacent to $\angle A$. List all angles opposite to $\angle A$.
   (iii) Define a pair of opposite sides and a pair of opposite angles without using the names of the vertices.

Solution: (i) A side is adjacent to AB if it shares an endpoint with AB. BC meets AB at B and DA meets AB at A. Therefore the adjacent sides are **BC and DA**. CD has no common endpoint with AB, so **CD** is the side opposite AB.

(ii) The angles next to angle A are angle B and angle D, since their vertices are joined to A by sides. The angle opposite A is angle C.

(iii) Two sides are opposite when they have no common endpoint. Two angles are opposite when their vertices are not joined by a side of the quadrilateral. This definition does not depend on the letters used to name its vertices.

Question 2

*2. You have used internal angles of quadrilaterals, but they too require an exact definition, just like how we gave one for a quadrilateral. Precisely define the internal angle of a quadrilateral at a given vertex. Your answer should work for a non-convex quadrilateral too. (**Hint:** use the opposite vertex as well.)

Solution: Let A be the vertex under consideration, B and D its adjacent vertices, and C its opposite vertex. The rays AB and AD create a smaller angle less than 180 degrees and a reflex angle greater than 180 degrees.

Case 1: A is outside triangle BCD. The inside of quadrilateral ABCD near A lies in the smaller angle between AB and AD. Hence the internal angle is angle DAB (less than 180 degrees).

Case 2: A is inside triangle BCD. The quadrilateral has a dent at A, and its interior occupies the reflex sector between AB and AD. Hence the internal angle equals **360 degrees minus angle DAB**. This definition works for convex quadrilaterals and for non-convex (dart-shaped) quadrilaterals. In a convex quadrilateral all four internal angles are smaller than 180 degrees.

Question 3

3. In a quadrilateral ABCD, suppose $AB \parallel DC$. Can ABCD be non-convex? What if we instead assume $AB = CD$? What if we instead assume $\angle A = \angle C$?

Solution: First condition, AB parallel DC: No, the quadrilateral cannot be non-convex. Its four vertices lie on two distinct parallel lines, with A and B on one and C and D on the other. A dent would require a vertex to lie inside the triangle formed by the other three; such a point would lie strictly between the parallel lines. The remaining vertex instead lies on one of the lines, so a dent is impossible.

Second condition, AB = CD: Yes, a non-convex example is A(0,0), B(6,0), C(3,2), D(3,8). Here AB = 6 and CD = 6. Because C lies inside triangle ABD, C is a reflex vertex and ABCD is non-convex.

Third condition, angle A = angle C: Yes. Take A(-2,0), B(0,1), C(2,0), D(0,4). This symmetric dart has equal angles at A and C, but B is the inward-pointing vertex. Thus equality of opposite angles alone does not force convexity.

Question 4

4. Consider three non-collinear points A, B, C and draw the lines AB, BC, CA. For every possible location of point D in the plane outside these lines, decide if ABCD is self-intersecting, non-convex, or convex. (**Hint:** the three lines divide the plane into 7 regions.)

Solution: The three lines AB, BC, CA divide the plane into seven open regions. Keep the order of vertices A-B-C-D fixed and examine where the fourth vertex D is placed.

Region inside triangle ABC: D is a dent, so ABCD is non-convex.

Region beyond side AC, away from B: all four vertices are extreme, and the four sides do not cross. ABCD is convex.

Region beyond AB, away from C: CD crosses AB. The 4-gon is self-intersecting.

Region beyond BC, away from A: AD crosses BC. Again it is self-intersecting.

Three regions beyond the vertices A, B and C: one of the existing vertices lies inside the triangle formed by the other three. Each case gives a non-convex quadrilateral.

Conclusion: One region gives a convex quadrilateral; two give self-intersecting figures; four give non-convex quadrilaterals. The regions lying on the lines are excluded by the question.

Question 5

5. Can a quadrilateral be both self-intersecting and non-planar?

Solution: No. For a four-sided figure to be self-intersecting, two opposite sides, say AB and CD, must meet at a point E that is not an endpoint. The complete lines AB and CD intersect at E, and two intersecting lines determine a plane. Because A and B lie on the first line and C and D lie on the second, all four vertices lie in that plane. Therefore the figure is planar. So a figure cannot be both a self-intersecting 4-gon and a non-planar 4-gon.

Exercise Set 12.2 (Solutions)

Question 1

1. True or false?
   (i) A parallelogram with a right angle is a rectangle.
   (ii) A rhombus with perpendicular diagonals is a square.
   (iii) If the diagonals of a parallelogram are equal, then it is a rectangle.

Solution: (i) True. Adjacent angles of a parallelogram are supplementary. If angle A = 90 degrees, then B = 180 - 90 = 90 degrees. Opposite angles are equal, so C = A = 90 degrees and D = B = 90 degrees. All four angles are right angles: it is a rectangle.

(ii) False. A rhombus already has diagonals that intersect at right angles; this does not make it a square. For example, a rhombus with angles 60 and 120 degrees has perpendicular diagonals but no right angle, so it is not a square.

(iii) True. In parallelogram ABCD assume AC = BD. Compare triangles ABC and DCB: AB = DC (opposite sides), BC = CB (common), and AC = DB (given). By SSS, the triangles are congruent. Thus angle ABC = angle DCB. These adjacent angles total 180 degrees, so each is 90 degrees. Hence ABCD is a rectangle.

Question 2

2. The diagonal AC of a parallelogram ABCD bisects $\angle A$. Show that it also bisects $\angle C$ and that ABCD is a rhombus.

Solution: Given: ABCD is a parallelogram and diagonal AC bisects angle A; hence angle BAC = angle CAD.

To show: AC bisects angle C, and ABCD is a rhombus.

Proof: Since AB is parallel to DC, transversal AC gives angle BAC = angle ACD. Since AD is parallel to BC, the same transversal gives angle CAD = angle ACB. Combining these with the given equality, angle ACD = angle ACB. Therefore AC also bisects angle C. Now, in triangle ADC, angle CAD = angle ACD. Sides opposite equal angles are equal, so **CD = AD**. Opposite sides of a parallelogram are equal: AB = CD and BC = AD. Consequently AB = BC = CD = DA, which proves that **ABCD is a rhombus**.

Question 3

3. The following questions examine converses of true properties. Answer them with Yes or No. If your answer is No, what extra condition can you add so that the answer becomes Yes?
   (i) If the diagonals of a quadrilateral ABCD bisect its angles, must ABCD be a rhombus?
   (ii) If the diagonals of a quadrilateral ABCD bisect each other at right angles, must ABCD be a rhombus?
   (iii) If the diagonals of a quadrilateral ABCD are of equal length, must ABCD be a rectangle? (In the ancient Indian study of quadrilaterals, equality of diagonals was considered significant. Quadrilaterals were first classified according to whether their diagonals were equal or not, before considering equality of sides.)

Solution: (i) Yes. Suppose the diagonals bisect the angles at all their endpoints. Since AC bisects A and C, triangles ABC and ADC have two pairs of equal angles and the included side AC common; ASA gives triangle ABC congruent to triangle ADC. Hence AB = AD and BC = CD. Similarly, using BD as an angle bisector at B and D makes triangles BAD and BCD congruent; therefore AB = BC and AD = CD. Thus all four sides are equal, so ABCD is a rhombus.

(ii) Yes. Diagonals that bisect each other make ABCD a parallelogram. If they also meet at 90 degrees at O, triangles AOB and AOD are congruent by SAS: OA is common, OB = OD and both included angles are right angles. Thus AB = AD. A parallelogram with equal adjacent sides is a rhombus.

(iii) No. An isosceles trapezium has equal diagonals but need not have four right angles. An extra sufficient condition is that the diagonals **also bisect one another**: then the shape is a parallelogram, and a parallelogram with equal diagonals is a rectangle.

Question 4

4. Let ABCD be a parallelogram with AB ≠ BC. Show that the pairwise intersection points of the four angle bisectors form the vertices of a rectangle (see Fig. 12.11). Why did we assume AB ≠ BC?

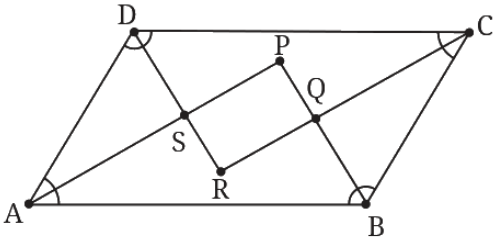


Fig. 12.11

Solution: Let the consecutive angle bisectors of parallelogram ABCD meet at P, Q, R, S as indicated in the textbook figure. Adjacent angles of a parallelogram are supplementary: for example angle A + angle B = 180 degrees. Their half-angles therefore add to 90 degrees. In the triangle formed by side AB and the bisectors from A and B, the angle between those bisectors must be 180 - 90 = 90 degrees. Exactly the same argument applies to the bisectors of B and C, C and D, and D and A. Thus the four interior angles of the figure PQRS are right angles. A quadrilateral with four right angles is a **rectangle**.

Why AB is not equal to BC: If AB = BC, the parallelogram is a rhombus. Its angle bisectors coincide with the diagonals, so they all meet at one point. The four intended vertices P, Q, R, S collapse together, and there is no ordinary rectangle.

Exercise Set 12.3 (Solutions)

Question 1

1. (i) If P, Q, R are the midpoints of sides AB, AC, BC respectively of ΔABC, show that ΔPQR is congruent to ΔQPA and to two other triangles which you should identify.
   (ii) Suppose someone erases ΔABC, leaving only ΔPQR on the paper. Can you reconstruct ΔABC from ΔPQR?

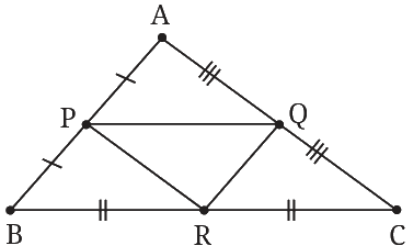


Fig. 12.22

Solution: (i) P and Q are midpoints of AB and AC, and R is midpoint of BC. By the Midpoint Theorem, PQ is parallel to BC and PQ = BC/2; PR is parallel to AC and PR = AC/2; QR is parallel to AB and QR = AB/2. In triangle PQR the three sides therefore have lengths BC/2, AC/2 and AB/2. Triangle APQ has AP = AB/2, AQ = AC/2 and PQ = BC/2. Triangle BPR has BP = AB/2, BR = BC/2 and PR = AC/2. Triangle CQR has CQ = AC/2, CR = BC/2 and QR = AB/2. All four triangles have the same three side lengths, so by **SSS** they are congruent: PQR, QPA, PBR and CQR.

(ii) Yes. Through P draw a line parallel to QR and through Q a line parallel to PR; their intersection is A. Similarly construct B and C using the remaining corresponding parallels. This reverses the midpoint construction, reproducing the original triangle uniquely.

Question 2

2. In ΔABC, let M and N be midpoints of AB and AC respectively. Let D be any point on BC. Show that MN bisects AD.

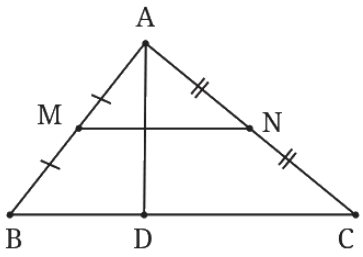


Fig. 12.23

Solution: Given: In triangle ABC, M and N are the midpoints of AB and AC. Point D lies anywhere on BC.

To prove: MN bisects AD.

Construction: Let X be the point at which segment AD meets MN.

Proof: By the Midpoint Theorem in triangle ABC, MN is parallel to BC. As BD lies on BC, MX is also parallel to BD. In triangle ABD, M is the midpoint of AB, and the line through M parallel to BD cuts AD at X. By the converse of the Midpoint Theorem, X is the midpoint of AD. Therefore **AX = XD**, so the segment MN bisects AD. This reasoning holds for every position of D on BC.

Question 3

3. In a quadrilateral ABCD, suppose AB ∥ DC and AB ≠ CD. Suppose G and H are the midpoints of AC and BD respectively. Prove that GH ∥ AB. (Why did we assume AB ≠ CD?)

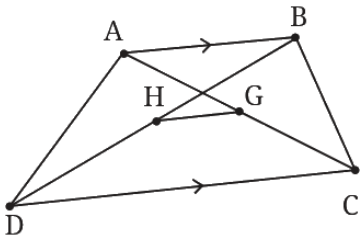


Fig. 12.24

Solution: Let M be the midpoint of AD. Let G be the midpoint of AC and H the midpoint of BD, as in the question. In triangle ADC, M and G are midpoints of AD and AC. By the Midpoint Theorem, MG is parallel to DC and has length DC/2. In triangle ADB, M and H are midpoints of AD and DB. Again by the Midpoint Theorem, MH is parallel to AB and has length AB/2. Because AB is parallel to DC, the lines MG and MH both pass through M and have the same direction; by uniqueness of a parallel through M, they are the same line. Hence M, G and H are collinear and **GH is parallel to AB and DC**. The condition AB is not equal to CD avoids the parallelogram case. If both AB = CD and AB parallel CD, the 4-gon is a parallelogram; its diagonals bisect each other, so G = H and GH is not a proper segment.

Question 4

4. Suppose the midpoints of sides AB, BC, CD and DA of a quadrilateral ABCD are P, Q, R and S respectively.
   (i) Show that PR and QS bisect each other.
   (ii) Show that if AC = BD, then PR and QS are perpendicular. Is the converse true?

Solution: (i) By the Midpoint Theorem, PQ is parallel to AC and SR is parallel to AC; hence PQ is parallel to SR. Similarly QR and PS are both parallel to BD. Therefore PQRS is a parallelogram (Varignon's Theorem). The diagonals of any parallelogram bisect each other, so PR and QS bisect one another.

(ii) The same midpoint construction gives PQ = AC/2 and QR = BD/2. If AC = BD, then PQ = QR. A parallelogram with two adjacent sides equal is a rhombus, and the diagonals of a rhombus meet at right angles. Hence PR is perpendicular to QS.

Converse: If PR and QS are perpendicular, parallelogram PQRS has perpendicular bisecting diagonals, so PQRS is a rhombus. Thus PQ = QR, or AC/2 = BD/2, giving **AC = BD**. Therefore the converse is true.

Question 5

5. Suppose PQRS is the Varignon parallelogram of ABCD.

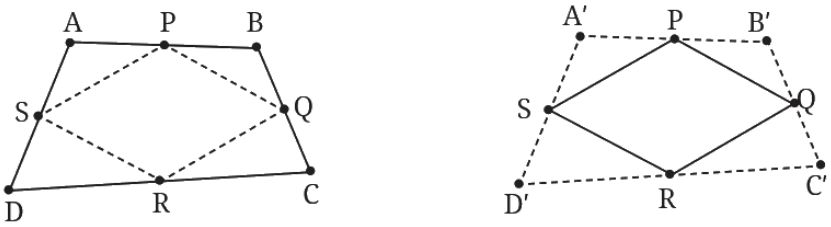


Fig. 12.25

(i) Copy only PQRS on another paper. Show how you will recreate a congruent copy A′B′C′D′ of ABCD from PQRS. This will be relevant when we return to study tilings at the end of this chapter. (**Hint:** How will you place vertex A′? How will you place B′, C′ and D′?)

(ii) There are multiple ways to construct A′B′C′D′ in (i). Justify why the quadrilateral A′B′C′D′ you constructed is congruent to ABCD. You may need to show why S is collinear with the points A′ and D′ that you constructed, and similarly for P, Q and R. (**Hint:** Use congruence of triangles, for example $\Delta SDR \cong \Delta SD'R$.)

(iii) Show that if PQRS is a square, then AC and BD are perpendicular and equal. Prove the converse.

Solution: (i) Construction: Copy the Varignon parallelogram PQRS. Choose a point A' so that S and P are the intended midpoints of A'D' and A'B'. Reflect A' in P to obtain B', then reflect B' in Q to obtain C', and reflect C' in R to obtain D'. The final requirement is that reflecting D' in S returns to A'. The parallelogram identity for PQRS ensures that these midpoint equations are consistent. Different starting choices of A' can produce different 4-gons with the same Varignon parallelogram.

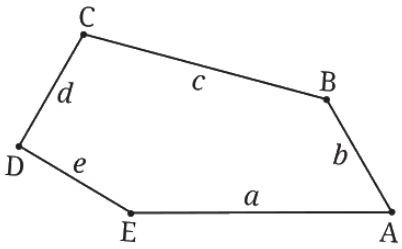
(ii) Why a congruent copy is possible: Choose A' to correspond to A in the given ABCD, maintaining the same displacement from S and P. The successive half-turns about P, Q, R recover B', C', D' with side lengths and included angles matching those of ABCD. Equalities of the resulting triangles can be shown by SAS; S is indeed the midpoint of A'D'. Hence this construction can reproduce a congruent copy.

(iii) Since PQ is parallel to AC and PQ = AC/2, and QR is parallel to BD and QR = BD/2, PQRS is a square exactly when its adjacent sides are equal and perpendicular. This is equivalent to **AC = BD and AC perpendicular to BD**. Both directions of the statement are proved.

Exercise Set 12.4 (Solutions)

Question 1

1. Justify why the plane cannot be tiled with a regular pentagon. (**Hint:** Read the first 3 sentences of 'Think and Reflect' in the section on tiling.) (There are many ways to tile the plane using a suitable irregular pentagon. The most recent method was found in 2015.)



$A = 60°$
$B = 135°$
$C = 105°$

$D = 90°$
$E = 150°$

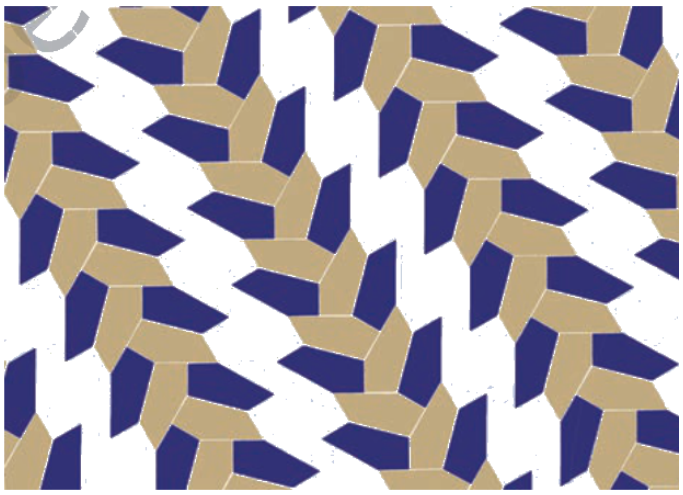
$a = 1$

$b = \frac{1}{2}$

$c = \frac{1}{\sqrt{2}(\sqrt{3}-1)}$

$d = \frac{1}{2}$

$e = \frac{1}{2}$



Solution: A regular pentagon has five equal interior angles. The angle sum is (5 - 2) x 180 = 540 degrees, so each interior angle equals **108 degrees**. When regular pentagons meet around a common vertex in a tiling, their angles must add to exactly 360 degrees. Three angles make 324 degrees, leaving a gap of 36 degrees; four angles make 432 degrees, an overlap. No whole number of 108-degree angles equals 360 degrees. Hence regular pentagons cannot tile the plane without gaps or overlaps in an edge-to-edge regular-pentagon tiling. Some irregular pentagons can tile because their angles need not all be 108 degrees.

Question 2

2. Draw a non-convex 4-gon DART. Show how we can tile the plane with copies of DART. Both methods that we discussed earlier will work. Which do you prefer?

Solution: Draw a planar non-convex quadrilateral DART with one inward vertex and trace many congruent copies on paper.

Method 1 (half-turns): Put down one copy. Select one of its sides, rotate a second identical copy by 180 degrees around the midpoint of that side, and place it against the first. Repeat across exposed sides. Each new copy fits its neighbour exactly along the selected side. The four interior angles of the non-convex quadrilateral still add to 360 degrees, so convex tips and reflex dents can meet with no gap at vertices. Continue the same local rule to extend the pattern.

Method 2 (Varignon grid): Join the side midpoints of DART to form its Varignon parallelogram. Tile the plane with this parallelogram, and place suitably oriented copies of DART in the grid; the remaining spaces are filled by congruent half-turned copies.

Preference: Method 1 is easier with physical cutouts. Method 2 provides a useful regular grid for understanding why the construction works.

End-of-Chapter Exercises (Solutions)

Question 1

1. Using a fact about parallelograms, show how to tile the plane using any given triangle. (**Hint:** Can you use the parallelogram tiling in the introduction?)

Solution: Take a triangle ABC. Make a congruent copy and rotate it by 180 degrees about the midpoint of side BC. The two triangles then share BC and their union is a parallelogram: the corresponding opposite sides are parallel and equal. A parallelogram can be repeated by translation in two independent side directions to tile the entire plane without gaps or overlaps. In each parallelogram tile we see exactly two copies of the original triangle. Therefore **every triangle tiles the plane**. The essential fact used is that a triangle together with its half-turned copy forms a parallelogram.

Question 2

2. Mark the midpoint of the line drawn on the paper (see Fig. 12.33), given that the horizontal lines are equally spaced. Justify your answer.

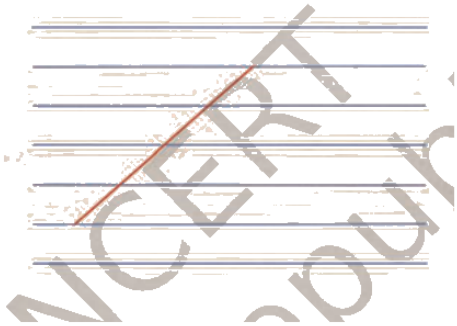


Fig. 12.33

Solution: The horizontal lines in Fig. 12.33 are equally spaced, and the oblique segment crosses them. Mark the intersection of the oblique segment with the horizontal line halfway between the lines through its endpoints. This is the line that leaves an equal number of horizontal gaps on the two sides.

Justification: A family of parallel, equally spaced lines cuts any transversal into equal segments. This follows from the intercept theorem, or by forming congruent right triangles between consecutive parallels. Since the marked point is two equal intervals from each endpoint (as shown in the figure), the lengths along the original sloping segment on either side are equal. Therefore the point is its **midpoint**.

Question 3

3. You know that the sum of angles of a quadrilateral is 360°, even for a non-convex quadrilateral. (Recall the proof.) Now consider a self-intersecting quadrilateral ABCD, where AB and CD intersect at point E. Show that $\angle A + \angle B + \angle C + \angle D < 360°$. Can you construct ABCD such that $\angle A + \angle B + \angle C + \angle D = 2°$?

Solution: Suppose the opposite sides AB and CD of the self-intersecting quadrilateral meet at E. The crossing divides the figure into triangles ADE and BCE. In triangle ADE, angle A + angle D = 180 degrees - angle AED. In triangle BCE, angle B + angle C = 180 degrees - angle BEC. The angles AED and BEC are vertically opposite, so each has the same size, say theta. Adding gives angle A + angle B + angle C + angle D = (180 - theta) + (180 - theta) = **360 degrees - 2 theta**. Because theta is positive, the sum is strictly less than 360 degrees. To make the sum 2

degrees, choose the crossing angle theta = 179 degrees. Then 360 - 2(179) = 2 degrees. Place the four endpoints along the two intersecting lines to form the corresponding crossed quadrilateral.

Question 4

4. In a parallelogram ABCD, two points P and Q are taken on diagonal BD such that DP = BQ (see Fig. 12.34). Show that APCQ is a parallelogram.

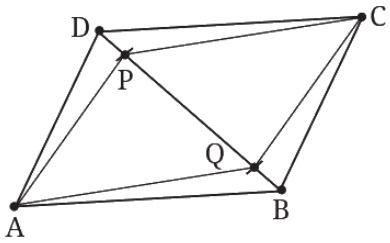


Fig. 12.34

Solution: Given: ABCD is a parallelogram. P and Q lie on diagonal BD and DP = BQ.

To show: APCQ is a parallelogram. Let O be the intersection of AC and BD. As diagonals of a parallelogram bisect each other, AO = OC and BO = OD. The points on BD appear in order D, P, O, Q, B in the illustrated configuration. Now OP = OD - DP and OQ = OB - BQ. Since OD = OB and DP = BQ, it follows that **OP = OQ**. Thus O bisects both AC and PQ, the two diagonals of APCQ. By the converse of the parallelogram-diagonal theorem, a quadrilateral whose diagonals bisect each other is a parallelogram. Hence **APCQ is a parallelogram**.

Question 5

5. A right-triangle shaped cutout of a paper is folded such that point A touches point B (Fig. 12.35). Show that the crease line can be used to find the midpoint of not only AB but also that of AC.

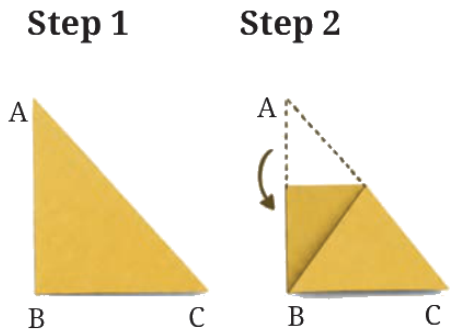


Fig. 12.35

Solution: Let the triangle be ABC, with angle B = 90 degrees and the paper folded so that point A coincides with B. Every point on a fold crease is equally far from A and B, so the crease is the perpendicular bisector of AB. If it cuts AB at M, then AM = MB: M is the midpoint of AB. Since the crease is perpendicular to AB and BC is also perpendicular to AB, the crease is parallel to BC. Let it meet AC at N. In triangle ABC, the line through the midpoint M of AB and parallel to BC bisects the third side AC by the converse of the Midpoint Theorem. Therefore **AN = NC**. The single crease locates the midpoints of both AB and AC.

Question 6

6. You saw how to use the Midpoint Theorem to divide a given triangle into 4 congruent triangles. Can we divide a triangle into 3 congruent triangles? This exercise shows us how to do that and more, provided we are allowed to cut and reassemble.

(i) Draw medians AP, BQ and CR of ΔABC, meeting in a common point M. Cut along each median to get 6 triangles. Show with justification how to assemble the 6 pieces into 3 congruent triangles! (**Hint:** Align ΔMPB and ΔMPC along equal sides PB and PC, matching P with itself and B with C.) Prove that you get a triangle. What are its side lengths?

*(ii) Repeat the procedure with each of the 3 assembled triangles you got in (i). Show that each of the resulting 9 triangles has sides $\frac{AB}{3}$, $\frac{BC}{3}$ and $\frac{AC}{3}$.

The procedure and the result above were unnoticed till 2014, when they were discovered by Lee Sallows, an amateur mathematician!

(iii) Naturally, we can next ask about the possibility of cutting a triangle and reassembling it into 2 congruent triangles. You were asked this question in the chapter *Measuring Space: Perimeter and Area*. Can you solve this now? (**Hint:** Use median AP in ΔABC and then a median of ΔAPB.)

For any two given figures with straight sides, we can ask whether one figure can be cut (using only straight cuts) into pieces and reassembled to make the other. This notion is called scissors congruence. The figures need not be in the plane. Research on such questions at a higher level is ongoing.

Solution: (i) Let AP, BQ and CR be the medians (P, Q, R are the midpoints of BC, CA, AB), meeting at the centroid M. The six small triangles are MPB, MPC, MQC, MQA, MRA and MRB. The centroid divides each median in the ratio 2 : 1, so MA = (2/3)AP, MB = (2/3)BQ, MC = (2/3)CR, and MP = AP/3, MQ = BQ/3, MR = CR/3.

Assembling: Take ΔMPB and ΔMPC. Since PB = PC, rotate ΔMPC through 180° about P. Then C falls on B, and M falls on a point M' with P the midpoint of MM'. The triangles MPB and M'PB now share the side PB, and M, P, M' lie on one straight line (P is the midpoint of MM'), so together they form one triangle MBM'. Its sides are MB = (2/3)BQ, BM' = CM = (2/3)CR and MM' = 2MP = (2/3)AP.

Do the same with the pair ΔMQC and ΔMQA (rotate about Q) and with the pair ΔMRA and ΔMRB (rotate about R). The three triangles formed have sides (2/3)AP, (2/3)BQ, (2/3)CR in each case, so they are congruent by SSS. Each is a genuine triangle (two small triangles joined along a side with the third vertices on a line through the midpoint), and together they use all six pieces, so the total area is the area of ΔABC. **Each assembled triangle has sides equal to two-thirds of the three medians.**

(ii) Let the sides of ΔABC be a = BC, b = CA, c = AB, and let its medians be m_a, m_b, m_c. By Apollonius' theorem, $m_a^2 = (2b^2 + 2c^2 - a^2)/4$, and similarly for m_b, m_c. The assembled triangle T in (i) has sides $(2/3)m_a$, $(2/3)m_b$, $(2/3)m_c$. Applying the same theorem to a triangle with sides m_a, m_b, m_c, its median to the side m_a satisfies $[2m_b^2 + 2m_c^2 - m_a^2]/4 = [(4a^2 + b^2 + c^2)/2 - (2b^2 + 2c^2 - a^2)/4]/4 = 9a^2/16$, so that median equals (3/4)a. Hence the medians of T are (2/3)(3/4)a = a/2, b/2, c/2.

Repeating the procedure on T gives triangles whose sides are two-thirds of these medians: (2/3)(a/2) = a/3, b/3, c/3. So each of the 9 triangles has sides **AB/3, BC/3 and AC/3**.

(iii) Let P be the midpoint of BC and cut ΔABC along the median AP into ΔABP and ΔACP. Let M be the midpoint of AB and cut ΔABP along its median PM into ΔAPM and ΔBPM. Rotate ΔBPM through 180° about M: B falls on A, and P falls on a point P' with M the midpoint of PP'. Because ∠AMP + ∠AMP' = ∠AMP + ∠BMP = 180°, the points P, M, P' are collinear, so ΔAPM and the rotated piece together form the triangle APP'.

Its sides are AP, AP' = BP = BC/2 and PP' = 2PM = AC (PM is the line joining the midpoints of AB and BC, so PM = AC/2). The triangle ACP has sides AP, CP = BC/2 and AC. By SSS, ΔAPP' ≅ ΔACP. So ΔABC has been cut into pieces that form two congruent triangles, each with half the area. Only straight cuts, translations and rotations were used.

Question 7

7. **A more general midpoint theorem and its converse.** In a quadrilateral ABCD, suppose AB ∥ DC. Recall that such ABCD is called a trapezium. Let E be the midpoint of AD. A line drawn through E intersects side BC at F.

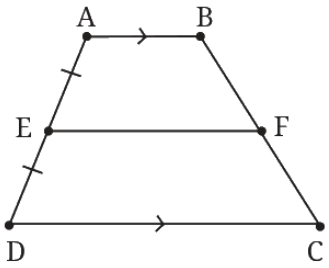


Fig. 12.36

(i) If EF ∥ AB, then show that F is the midpoint of BC. Conclude that EF = $\frac{(AB + CD)}{2}$.

(ii) If F is the midpoint of BC, then show that EF ∥ AB.

There are at least two ways to solve (ii). Do it directly by using the midpoint M of BD and showing that EM and FM are the same lines. Or use (i) along with the same idea used in the second proof of the converse of the Midpoint Theorem. Which way do you think is simpler?

Solution: Given: ABCD is a trapezium with AB parallel DC, E the midpoint of AD, and EF meeting BC at F.

(i) Assume EF is parallel to AB (and thus DC). Draw diagonal AC and let G be its intersection with EF. In triangle ADC, E is the midpoint of AD and EG is parallel to DC. The converse of the Midpoint Theorem gives AG = GC; the Midpoint Theorem gives EG = DC/2. In triangle ABC, G is now midpoint of AC and GF is parallel to AB. Hence F is midpoint of BC and GF = AB/2. Therefore **EF = EG + GF = (DC + AB)/2**.

(ii) Assume both E and F are side midpoints. Let M be midpoint of diagonal BD. In triangle ABD, segment EM joins midpoints, so EM is parallel to AB. In triangle DBC, segment MF joins midpoints, so MF is parallel to DC. Since AB is parallel to DC, EM and MF are on the unique line through M parallel to the bases. Thus E, M, F are collinear and **EF is parallel to AB**.

Question 8

8. The diagonals AC and BD of a parallelogram ABCD intersect at O. A line through O meets AB and CD at points P and Q respectively. Show that O is the midpoint of PQ. (Multiple proofs are possible. Which is the simplest?)

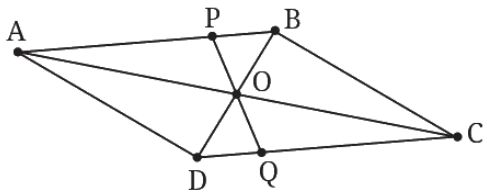


Fig. 12.37

Solution: Let O be the intersection of diagonals AC and BD in parallelogram ABCD, and let the line through O intersect AB at P and CD at Q. Opposite sides AB and CD are parallel, so angle OAP = angle OCQ (alternate interior angles along AC). Also angle AOP = angle COQ (vertically opposite angles). As the diagonals of a parallelogram bisect each other, OA = OC. Therefore triangles OAP and OCQ are congruent by **ASA**. Corresponding sides are equal, so OP = OQ. Since P, O and Q lie on the same straight line, **O is the midpoint of PQ**.

Question 9

*9. ABCD is a trapezium with parallel sides AD = 3 cm and BC = 5 cm. E and F are the midpoints of the non-parallel sides. Find the ratio of the areas of the 4-gons AEFD and EBCF.

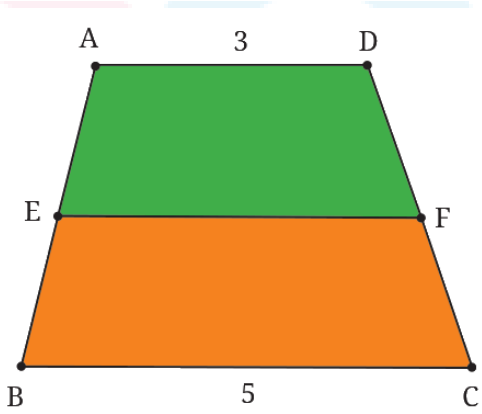


Fig. 12.38

Solution: Let the trapezium height be h. Its parallel sides have lengths AD = 3 cm and BC = 5 cm. E and F are midpoints of the non-parallel sides, so EF is the mid-segment: EF = (3 + 5)/2 = **4 cm**. The line EF lies midway between the two bases, so each of the smaller trapezia has height h/2. Area of AEFD = (1/2)(AD + EF)(h/2) = (1/2)(3 + 4)(h/2) = 7h/4. Area of EBCF = (1/2)(EF + BC)(h/2) = (1/2)(4 + 5)(h/2) = 9h/4. The area ratio is (7h/4):(9h/4) = **7:9**. The height cancels, so no numerical height is needed.

Question 10

10. Consider 4 points A, B, C, D in the plane with no three collinear. Answer the following questions. Some answers may require you to consider different cases, depending on how the points are positioned in the plane.
    (i) How many different quadrilaterals do they form if the quadrilateral is allowed to be self-intersecting or non-convex?
    (ii) How many of these quadrilaterals are self-intersecting? How many are convex?

Solution: (i) Label the four distinct points A, B, C, D. A quadrilateral is determined by choosing a cyclic ordering of these points. There are 4! = 24 linear orders; rotations of the same circuit give the same figure (divide by 4) and reversing the direction gives the same sides (divide by 2). Thus there are **24/(4 x 2) = 3 different 4-gons** if self-intersecting and non-convex arrangements are permitted.

(ii) Case 1: four points in convex position. Connecting them in their boundary order gives one convex quadrilateral. The two other cyclic orders cross a pair of opposite sides, so there are **2 self-intersecting** quadrilaterals.

Case 2: one point lies inside the triangle formed by the other three. No ordering is convex, and there is no crossing of opposite sides; the four points give **3 non-convex** quadrilaterals. These two positional cases exhaust the possibilities when no three are collinear.

Question 11

11. Suppose P is a point on side AB of ΔABC and the line through P parallel to BC meets AC in point Q. For parts (i) to (iii), assume AP = 1, AQ = $\sqrt{5}$, and see Fig. 12.39.

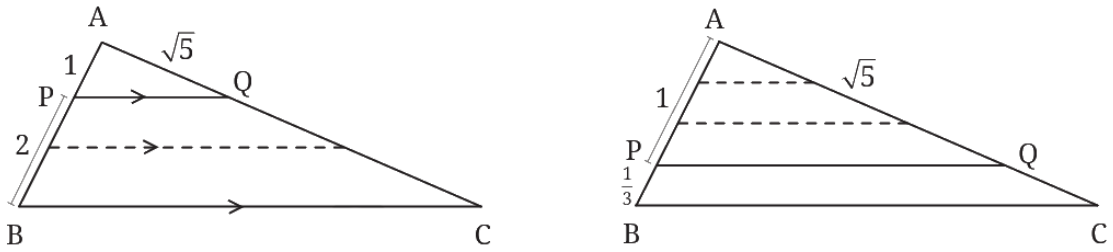


Fig. 12.39

(i) If PB = 2 find QC. (**Hint:** See the dotted line)

*(ii) If PB = $\frac{1}{3}$ find QC. (**Hint:** See the dotted lines)

*(iii) If PB = $\frac{2}{3}$ find QC.
(**Hint:** Divide both PB and AP suitably.)

*(iv) Show that if $\frac{AP}{PB}$ is a rational number, then $\frac{AP}{PB} = \frac{AQ}{QC}$. In Grade 10, you will prove this equality without assuming $\frac{AP}{PB}$ is rational. That will require a new idea.

Solution: Because P is on AB and PQ is parallel to BC, parallel lines cut the sides AB and AC in the same ratio. Thus AP/PB = AQ/QC. Here AP = 1 and AQ = sqrt(5).

(i) If PB = 2, then 1/2 = sqrt(5)/QC, so **QC = 2 sqrt(5)**.

(ii) If PB = 1/3, then 1/(1/3) = sqrt(5)/QC, hence **QC = sqrt(5)/3**.

(iii) If PB = 2/3, then 1/(2/3) = sqrt(5)/QC, hence **QC = 2 sqrt(5)/3**.

(iv) If AP/PB is rational, write it m/n with positive integers m,n. Divide the length AB into m+n equal small pieces, with AP consisting of m pieces and PB of n pieces. Draw a line parallel to BC at each division mark. Equal intercepts are cut on AC, with AQ containing m equal pieces and QC containing n equal pieces. Therefore **AP/PB = AQ/QC**, first for rational ratios. The general result is the Basic Proportionality Theorem studied later.

Question 12

12. (i) Suppose ABCD is a parallelogram and M, N are midpoints of AB and CD respectively. Show that segments DM and BN trisect segment AC.
(ii) Use part (i) to find a procedure to trisect any given segment PQ. Find two other ways to trisect PQ, one using Exercise 11 above and a third using the Centroid Theorem.

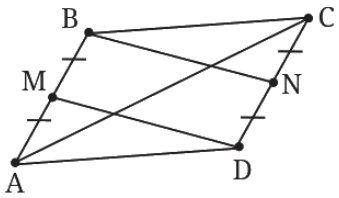


Fig. 12.40

Solution: (i) Let M and N be the midpoints of the opposite sides AB and CD of the parallelogram ABCD. Since AB is parallel and equal to CD, MB is parallel and equal to DN, so MBND is a parallelogram and DM is parallel to NB.

Let DM meet the diagonal AC at X and NB meet AC at Y. In ΔABY, M is the midpoint of AB and MX is parallel to BY, so X is the midpoint of AY (converse of the Midpoint Theorem): AX = XY. In ΔCDX, N is the midpoint of CD and NY is parallel to DX, so Y is the midpoint of CX: CY = YX. Therefore AX = XY = YC, and DM and BN **trisect** the diagonal AC.

(ii) To trisect a given segment PQ, use PQ as the diagonal of a parallelogram and apply (i): the two lines from the opposite vertices to the midpoints of the opposite sides cut PQ into three equal parts. A second method is to draw a ray from P, mark three equal lengths on it, join the last mark to Q, and draw parallels to this line through the other two marks. A third method uses the centroid: build a triangle in which PQ is a median. The centroid divides PQ in the ratio 2 : 1, so it is one trisection point, and the midpoint of the longer part is the other.

Question 13

13. Is the Midpoint Theorem for Quadrilaterals (Theorem 9) true when the quadrilateral is non-convex? How about when it is self-intersecting? Experiment and check. The pictures may look strange, but the theorem appears to be true.
(i) Can you explain why this is so? See the geometric reasoning in the text.
(ii) There is one exception: In a very special case the Varignon parallelogram becomes a single segment. When will this happen?
*(iii) Does your reasoning apply even when ABCD is non-planar?

Solution: (i) Let P,Q,R,S be the midpoints of AB, BC, CD, DA respectively. In triangle ABC, PQ is parallel to AC and equals AC/2. In triangle ADC, SR is parallel to AC and equals AC/2. So PQ and SR are equal and parallel. Similarly, in triangles BCD and BAD, QR and PS are equal and parallel to BD. Therefore PQRS is a parallelogram. This argument uses only triangle midpoints, not convexity of ABCD, so it remains valid for a non-convex or a self-intersecting 4-gon.

(ii) If diagonals AC and BD are parallel (the special degenerate arrangement), the sides of the Varignon figure that are parallel to AC and BD lie along the same direction, and the four midpoints lie on one straight line. The parallelogram collapses into a segment.

(iii) The argument still works when ABCD is non-planar: both opposite sides of PQRS are equal and parallel to the respective diagonal directions. Two intersecting lines through P determine a plane, and the remaining midpoint vertices lie in that plane. Hence the midpoint figure is planar and is a parallelogram even for a skew 4-gon.

Question 14

14. Let P, Q, R, S be four points on sides AB, BC, CD and DA respectively of a quadrilateral ABCD. Suppose PQRS is a parallelogram. Must P, Q, R and S be midpoints of the respective sides? This can be considered a possible converse question to Theorem 9.

Solution: No. The Midpoint Theorem for Quadrilaterals says that the side midpoints form a parallelogram, but this condition does not work in reverse. As a counterexample, take a square ABCD of side length 4. On AB, BC, CD and DA mark P, Q, R and S at the same fraction (say one quarter) of the way around the boundary, rather than halfway. The four small triangles cut off at the corners are congruent by SAS, and the resulting inner quadrilateral PQRS is a square, so it is a parallelogram. Yet AP = BQ = CR = DS = 1, while each whole side is 4, so none of P,Q,R,S is a midpoint. Therefore a parallelogram PQRS does not force the points to be side midpoints.

Question 15

15. (i) Complete the following proof of the Midpoint Theorem. Extend segment PQ beyond Q until point S. By how much should we extend PQ?

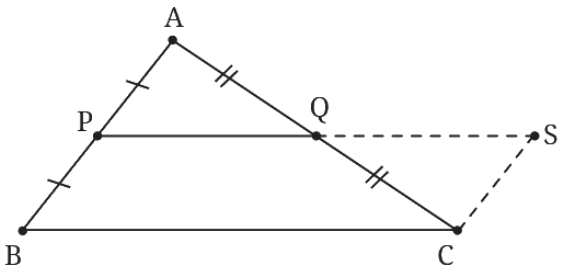


Fig. 12.41

It would be good to be able to prove that $\Delta APQ \cong \Delta CSQ$. (Why?) Use this as a guide to specify the location of S and then complete this proof.

(ii) Give a similar proof of the converse of the Midpoint Theorem (Theorem 7). Start by extending PQ up to a suitable point S.

Solution: (i) In triangle ABC let P and Q be midpoints of AB and AC. Extend PQ beyond Q to S such that **QS = PQ**. Then AQ = QC, PQ = QS, and the angles AQP and CQS are vertically opposite. Thus triangles APQ and CSQ are congruent by SAS. Their corresponding sides give AP = CS and parallel directions AP parallel CS. Since AP = PB, we have PB = CS and PB is parallel CS. Hence PBCS is a parallelogram by the equal-and-parallel-sides test. Therefore PS is parallel BC and PS = BC. But PS = PQ + QS = 2PQ. Consequently **PQ is parallel BC and PQ = BC/2**, proving the Midpoint Theorem.

(ii) Conversely, let P be midpoint of AB and let the line PQ through P be parallel to BC, meeting AC at Q. Extend PQ to S such that QS = PQ. Repeat the angle and side argument: because the constructed parallel lines give the matching angles and P is the midpoint of AB, one obtains triangles APQ and CSQ congruent, forcing AQ = QC. Thus Q is the midpoint of AC. The same parallelogram gives PQ = BC/2.

Question 16

*16. Review all the properties of a rhombus/rectangle/square that you proved in Grade 8. Formulate a converse of each. Decide if the converse is true. There are many possibilities here!

Solution: Review the properties and write a converse for each, rather than treating every converse as automatically true.

Rhombus: A rhombus has perpendicular bisecting diagonals. Converse: if diagonals are perpendicular and bisect each other, the 4-gon is a parallelogram with equal adjacent sides, hence a rhombus (true). But perpendicular diagonals alone are not sufficient: a kite can have perpendicular diagonals and unequal sides. A rhombus also has diagonals that bisect its vertex angles; if both diagonals bisect the vertex angles, the converse is true.

Rectangle: The diagonals of a rectangle are equal. Equal diagonals alone do not prove rectangle: an isosceles trapezium is a counterexample. Add that the diagonals bisect each other; then the figure is a parallelogram with equal diagonals and therefore a rectangle.

Square: A square has equal, perpendicular diagonals that bisect each other. The converse with all three conditions is true, because equal bisecting diagonals give a rectangle, and perpendicularity gives a rhombus. Either equality or perpendicularity alone is insufficient.

Question 17

*17. Show that the sum of angles of a non-planar quadrilateral is always less than 360°. Can you find a non-planar quadrilateral ABCD for which $\angle A + \angle B + \angle C + \angle D = 2°$?
(**Hint:** Think of a diagonal, say AC, as a hinge around which triangles ABC and ADC can rotate.) What happens to each angle of ABCD as you do this rotation?

Solution: Join two triangles ABC and ADC along the common diagonal AC to form a quadrilateral ABCD. In the flat (planar) position the four interior angles add up to 360°.

Now hold ΔADC fixed and turn ΔABC about the hinge AC, out of the plane. The angles of the two triangles do not change, so the angles at B and D stay the same. At the vertex A, the angle between AB and AD is now less than $\angle BAC + \angle CAD$ (in any solid angle at a vertex, one face angle is less than the sum of the other two, with equality only when the three are in one plane). The same is true at C. So the sum of the four angles is strictly less than 360°.

An example with sum 2°: Take two congruent isosceles triangles ABC and ADC on the common base AC with the apex angles $\angle B = \angle D = 0.5°$. In the flat position, ABCD is a rhombus with the angle sum 360°. As the hinge closes, the angles at A and C decrease continuously from 179.5° down to 0°, while the angles at B and D stay 0.5°. When the two triangles lie on top of each other, the sum is 0.5° + 0.5° = 1°. Since the sum changes continuously from 360° to 1°, there is a position of the hinge at which the sum is exactly 2°.

Question 18

*18. Let us see a third method to tile the plane using a 4-gon.

Focus on only two *coloured copies* of SOME sharing a vertex. What do you see? It appears that each 4-gon is just a shifted copy of the other. Let us see exactly how. Draw two copies of SOME as shown in Fig. 12.42 so that the diagonals EO and E′O′ are collinear with O = E′. Now place a cutout of one copy on top of SOME with diagonal EO drawn on it. Slide this cutout so that segment EO moves along EO′ until EO matches E′O′. Verify that your cutout exactly matches S′O′M′E′.

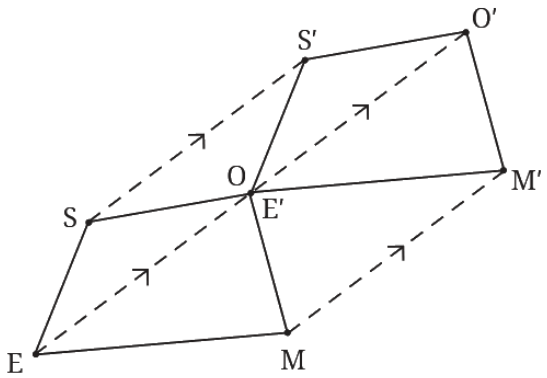


Fig. 12.42

(i) Prove the exact match of SOME with S′O′M′E′ using four parallelograms.

(ii) Follow the described procedure along each diagonal of the starting coloured 4-gon in both directions to get 4 new 4-gons. Using these 4 copies, repeat the procedure 4 more times to place 4 new copies, resulting in a 3 by 3 grid of 9 copies exactly like the 9 coloured copies in Fig. 12.42. These 9 copies meet each other only at vertices and once again the blank spaces among them also form 4-gons congruent to SOME!

Solution: (i) Draw two labelled copies SOME and S'O'M'E' as in Fig. 12.42, with their corresponding diagonals along one line. A translation through vector EE' sends E to E', O to O', S to S' and M to M'. For each pair of corresponding vertices, the connecting segments are equal and parallel. Each quadrilateral formed by one side of SOME, its translated side and the connectors between corresponding endpoints has one pair of opposite sides equal and parallel. By the parallelogram test it is a parallelogram. In this way four parallelograms establish that all four sides and angles match exactly under the shift.

(ii) Shift the original copy forward and backward along each of its two diagonal directions to create four neighbours. Apply the same translations again to obtain the indicated 3 by 3 grid of nine congruent copies. Neighbouring tiles in this grid touch only at vertices. At each resulting opening, fit a copy obtained by a half-turn; matching opposite sides and the angle sum 360 degrees show that it fills the opening without overlap. Continuing the translations fills the plane, giving the third tiling method.

Question 19

*19. **Is there a 4-gon with given side lengths?**

(i) Recall the following fact about triangles and check it by construction. For given positive numbers a, b, c, is there a triangle whose sides have these lengths? The answer is Yes exactly when the sum of any two numbers is greater than the third. If we arrange the numbers in increasing order (suppose $a \le b \le c$), then this amounts to requiring $a + b > c$. (**Hint:** Start by drawing a segment of length c.)

(ii) Suppose a 4-gon has 2, 5, 11 as three side lengths. Can the length of the fourth side be 100? Can it be 10? Can it be 1? What are the possible lengths of the fourth side?

(iii) For given positive numbers a, b, c, d, how will you decide if there is a 4-gon whose sides have these lengths?

Solution: (i) For three positive lengths a,b,c, arrange them so a is no greater than b and b is no greater than c. A non-degenerate triangle exists exactly when **a + b is greater than c**. Construct a segment of length c and draw arcs of radii a and b centred at its endpoints. The arcs cross at a point off the base precisely when a+b>c; joining this point to the endpoints makes the triangle.

(ii) Suppose the fourth side of a quadrilateral with other sides 2,5,11 has length x. The longest side must be less than the sum of the other three. From 11 < 2+5+x we get x > 4. From x < 2+5+11 we get x < 18. Therefore the possible lengths are **4 < x < 18**. In particular, x=100 is impossible, x=10 works, and x=1 is impossible.

(iii) For arbitrary positive lengths a,b,c,d, arrange them in increasing order. A non-degenerate quadrilateral exists if and only if the largest length is smaller than the sum of the other three. Necessity follows by going around the other three sides between endpoints of the longest side. Sufficiency follows by choosing a suitable diagonal and constructing two triangles on opposite sides of it whose combined boundary has the required four lengths.

Question 20

*20. **Counting diagonals of a polygon.**

(i) How should we define a diagonal of an n-gon? How many diagonals does an n-gon have? Make a table for small values of n. A 3-gon has no diagonals. A 4-gon has 2. How many diagonals does a 5-gon have? A 6-gon? Try to find a pattern and guess the answers for $n = 7$ and $n = 8$. Check your guesses by systematic counting.

(ii) Can you guess a formula for the number of diagonals? How many diagonals get added when we increase the number of sides by 1? Can you now justify why the formula you guessed is true for all n?

Solution: (i) A diagonal of a polygon joins two vertices that are not adjacent along its boundary. For n = 3, 4, 5, 6, 7, 8, the numbers of diagonals are respectively **0, 2, 5, 9, 14, 20**. This is the pattern obtained by counting chords from each vertex and excluding the two boundary neighbours.

(ii) From any vertex of an n-gon, one can draw diagonals to n-3 vertices (not to itself or its two neighbours). Summing over all n vertices gives n(n-3), but every diagonal has two endpoints and has been counted twice. Thus **D(n) = n(n-3)/2**. When n increases to n+1, the extra number is D(n+1)-D(n) = [(n+1)(n-2)-n(n-3)]/2 = **n-1**. This agrees with the table and justifies the formula for every n at least 3.

Question 21

*21. **Sum of angles of a polygon.** What is the sum of angles of a (planar non-self-intersecting) n-gon? We know that the answer is 180° for n = 3 and 360° for n = 4. Find the next few values. Then find a formula in terms of n and prove it.

Solution: The total interior-angle sum of an ordinary planar non-self-intersecting polygon is **(n-2) x 180 degrees**. Check the first values: a triangle (n=3) gives 180 degrees; a quadrilateral gives 360 degrees; a pentagon gives 540 degrees; a hexagon gives 720 degrees; a heptagon gives 900 degrees; an octagon gives 1080 degrees.

Proof for a convex polygon: Choose one vertex and draw all diagonals from it to non-neighbouring vertices. These divide the polygon into exactly n-2 triangles, whose interiors do not overlap. The sum of the triangle angles is therefore (n-2) x 180 degrees. For a non-convex simple polygon, divide it into triangles by repeatedly cutting off an ear or by using an appropriate triangulation. The number of triangles is still n-2, so the formula also holds there.

Question 22

*22. **Multiple converses to a theorem.** Let us see how multiple statements can be considered converses to the Midpoint Theorem and how Theorem 7 is one of them.

To formulate a converse we should express the original statement in "If ... then ..." form. For a complex statement, there may be multiple ways to do that. The Midpoint Theorem starts with ΔABC and points P and Q on sides AB and AC respectively.

The theorem has two assumptions and two conclusions, which we have named for further discussion.

Assumptions:	(P MID) P is the midpoint of AB, and	(Q MID) Q is the midpoint of AC.
Conclusions:	(PRLL) PQ ∥ BC, and	(HALF) PQ = $\frac{BC}{2}$.

Let us use these four named conditions to discuss various possible statements.

(i) A natural converse of the Midpoint Theorem would be: "If (PRLL) and (HALF) are true, then (P MID) and (Q MID) are true." Write out this statement fully. Experiment and see that it seems to be true! Prove the statement. (**Hint:** Try to run a proof of the Midpoint Theorem backwards.)

(ii) To see Theorem 7 as a converse of the Midpoint Theorem, we first write the Midpoint Theorem as follows: "Suppose P is the midpoint of side AB of ΔABC and Q is a point on side AC. If Q is the midpoint of AC then PQ ∥ BC." (Also PQ = $\frac{BC}{2}$, but let us set that aside for now.) Write the converse of the second sentence in quotes by keeping the first sentence the same. Verify that Theorem 7 is what you get! (And PQ = $\frac{BC}{2}$ also follows.) Now write Theorem 7 in terms of the four named conditions.

(iii) The discussion so far suggests that it is reasonable to take two of the four listed statements as assumptions and ask if the other two are true. Verify that only one such combination remains to be examined. "If (P MID) and (HALF) are true, then can we conclude (Q MID) and/or (PRLL)?" Write this out in words. This is a precise version of question (1) stated before Theorem 7. Can you answer it?

In the text, why did we focus only on Theorem 7 among the multiple possibilities? Because Theorem 7 turns out to be particularly useful. It is still important to pursue various avenues while exploring a subject: often it is only after examining multiple possibilities that we can judge what is most valuable.

Solution: Given: In triangle ABC, P is on AB and Q is on AC. Write P MID: P is midpoint of AB; Q MID: Q is midpoint of AC; PRLL: PQ is parallel BC; HALF: PQ = BC/2.

(i) The converse using both conclusions is: If PQ is parallel to BC and PQ = BC/2, then P and Q are midpoints of AB and AC. To prove it, use the line through C parallel AB and meet the extended PQ at R. Parallelism makes PBCR a parallelogram, with PR = BC = 2PQ, so Q is midpoint of PR. Comparing triangles APQ and CRQ using equal corresponding angles and PQ = QR gives congruence and then AP = PB, AQ = QC. Hence both points are midpoints.

(ii) Keeping P MID fixed, the converse says: If P is the midpoint of AB and PQ is parallel to BC, then Q is midpoint of AC. This is Theorem 7, which also gives HALF. In symbols, **(P MID and PRLL) implies (Q MID and HALF)**.

(iii) It is not enough to know P MID and HALF without parallelism. A point Q on AC can have distance PQ = BC/2 without being its midpoint in a suitable scalene triangle. For example, choose A=(0,0), B=(3,2), C=(4,0), P=(1.5,1)

and Q=(1,0). Here P is midpoint of AB and PQ = sqrt(5)/2 = BC/2, but Q is not midpoint of AC and PQ is not parallel to BC. Thus that remaining proposed converse is **false**.

Question 23

*23. **Validity of tiling methods.** Show using reasoning that each of the three tiling methods we saw produces a tiling of the plane using the given 4-gon. You have to prove that when we place new copies using any of the three procedures, the entire plane is covered with no gaps and no overlaps. First, think carefully about what it means to prove this. Then find a proof for each procedure.

Solution: We need to justify three things for every construction: neighbouring copies match along their shared boundaries, interiors do not overlap, and the pattern can be extended indefinitely to cover the whole plane.

Half-turn method: A 180-degree rotation about the midpoint of a side sends a copy to its neighbour, preserving lengths and angles. The common side is matched exactly. Around a new meeting point the four original vertex angles appear once each and sum to 360 degrees, so there is neither a hole nor an overlap. Repeating the rule outward creates every new ring of tiles.

Varignon-grid method: The four side midpoints form a parallelogram by the Midpoint Theorem. Translating that parallelogram makes an ordinary gap-free lattice. The quadrilateral copies occupy chosen cells, and the remaining gaps are congruent half-turned copies; the shared midpoint and congruence arguments ensure a perfect fit.

Diagonal-translation method: Translating the original shape by two independent diagonal displacement vectors produces a lattice of matching copies. The unoccupied cells are congruent to a half-turned original. Translational copies cannot change area or overlap along corresponding boundaries. The repeated lattice covers the plane. All three methods therefore produce tilings by copies of the given 4-gon.

Question 24

*24. What fraction of the square is shaded?

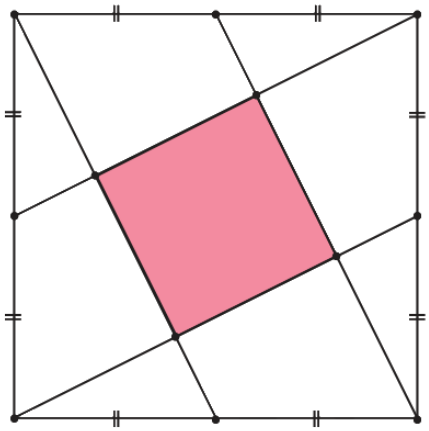


Fig. 12.43

Solution: Let the outer square in Fig. 12.43 have side length s. The marked points split the sides as shown, and the internal slant lines form the shaded central quadrilateral. Opposite slant sides are parallel and the corner constructions are symmetric, so the shaded central figure is a square. For one pair of parallel slant lines, the central region lies between two parallel strips. By the equal-mark and similar-triangle relations shown in the figure, the perpendicular separation of those lines is **s/sqrt(5)**. This distance is the side of the shaded square. Hence its area is (s/sqrt(5)) squared = s-squared/5. The area of the outer square is s-squared. Therefore the required fraction is **(s-squared/5)/(s-squared) = 1/5**. The result does not depend on the size of the outer square.