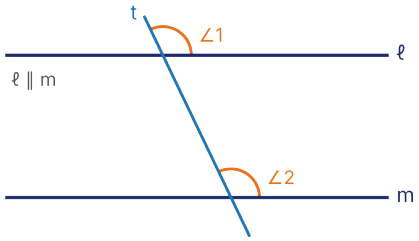
CHAPTER: 9 - Propositions and Their Converses

Exercise Set 9.1 (Solutions)

For Questions 1–12, write the converse of each statement. Decide separately whether the original and the converse are true; prove a true statement, or give a counterexample for a false one.

1. A transversal intersects two lines. Examine the proposition: parallel lines produce equal corresponding angles. Form and test its converse.



Solution:

Converse: If a transversal makes equal corresponding angles with two lines, then the two lines are parallel.

Proposition: True. This is the corresponding-angles property of parallel lines.

Converse: True. Suppose ∠1 = ∠2 but ℓ and m are not parallel. Then they meet at a point R, and together with the transversal they form a triangle. In that triangle, one of the two equal angles is an exterior angle and the other is an interior opposite angle. An exterior angle of a triangle equals the sum of the two interior opposite angles, so the angle at R would have to be 0°, which is impossible. Hence ℓ and m never meet, that is, $\ell \parallel$ m.

2. For a quadrilateral, consider the statement that being a square ensures all four angles have equal measures. Write and assess the converse.

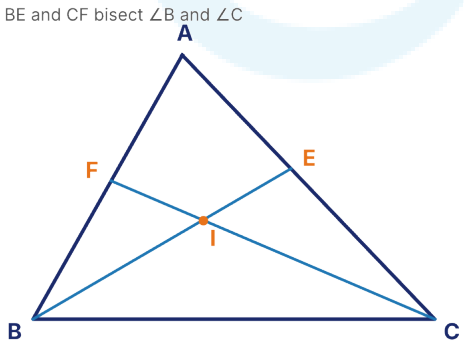
Solution:

Converse: A quadrilateral whose four angles are equal must be a square.

Proposition: True. Every angle of a square is 90°, so all four angles are equal.

Converse: False. A rectangle of 4 cm × 2 cm has four equal angles (each 90°), but its sides are not all equal, so it is not a square.

3. In triangle ABC, the internal angle bisectors from B and C cross at I. The bisector through B meets AC at E, and the bisector through C meets AB at F (Fig. 9.1). Test: AB = AC implies IE = IF; also test the reverse implication.



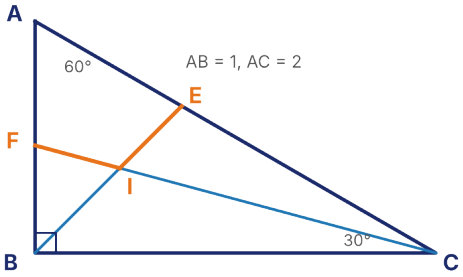
Solution:

Proposition: True. Let AB = AC. Then ∠B = ∠C, so their halves are equal: ∠IBC = ∠ICB = ∠IBF = ∠ICE. In ΔIBC the angles at B and C are equal, so IB = IC. Now compare ΔIBF and ΔICE: ∠IBF = ∠ICE, IB = IC, and ∠BIF = ∠CIE (vertically opposite angles). So ΔIBF ≅ ΔICE by ASA, and therefore IF = IE.

Converse: False. Take a right triangle with ∠A = 60°, ∠B = 90° and ∠C = 30°. Put B = (0, 0), A = (0, 1) and C = (√3, 0). Then AB = 1 and AC = 2, so AB ≠ AC. We now show that IE = IF in this triangle.

- The inradius of a right triangle is r = (AB + BC − AC)/2 = (√3 − 1)/2, so I = (r, r).
- The bisector of ∠B is the line y = x. It meets AC (the line x/√3 + y = 1) at E = ((3 − √3)/2, (3 − √3)/2).
- The bisector of ∠C makes an angle of 15° with CB, and tan 15° = 2 − √3. It meets AB (the y-axis) at F = (0, √3(2 − √3)) = (0, 2√3 − 3).

- $IE^2 = 2[(3 - \sqrt{3})/2 - (\sqrt{3} - 1)/2]^2 = 2(2 - \sqrt{3})^2 = 14 - 8\sqrt{3}$.
- $IF^2 = [(\sqrt{3} - 1)/2]^2 + [(2\sqrt{3} - 3) - (\sqrt{3} - 1)/2]^2 = (2 - \sqrt{3})/2 + (52 - 30\sqrt{3})/4 = 14 - 8\sqrt{3}$.



So IE = IF (about 0.38), while AB ≠ AC. Hence the converse is false.

Why this works: ∠BIC = 90° + ∠A/2 = 120°, so ∠EIF = 120° and ∠EAF + ∠EIF = 180°. Thus A, F, I, E lie on one circle. AI bisects ∠A, so the chords IF and IE subtend equal angles at A, and IF = IE whenever ∠A = 60°.

4. For any numbers a, x and y, if x = y then a + x = a + y. Is the converse valid?

Solution:

Converse: If a + x = a + y, then x = y.

Both statements are true. If x and y are the same number, adding the same number a to both gives equal results. Conversely, subtracting a from both sides of a + x = a + y gives x = y.

5. If a and b are perfect-square integers, their product ab is a perfect square. Investigate the converse.

Solution:

Converse: If ab is a perfect square, then each of a and b is a perfect square.

Proposition: True. If $a = p^2$ and $b = q^2$, then $ab = p^2q^2 = (pq)^2$.

Converse: False. Take a = 2 and b = 8. Then $ab = 16 = 4^2$ is a perfect square, but neither 2 nor 8 is a perfect square.

6. For real numbers x and y, consider: if x = y then $x^2 = y^2$. Give the converse and test both.

Solution:

Converse: If $x^2 = y^2$, then x = y.

Proposition: True. Squaring two equal numbers gives equal results.

Converse: False. Take x = 3 and y = −3. Then $x^2 = y^2 = 9$, but x ≠ y.

7. For real numbers x and y, investigate: equality x = y implies $x^3 = y^3$; also investigate the converse.

Solution:

Converse: If $x^3 = y^3$, then x = y.

Proposition: True. Cubing two equal numbers gives equal results.

Converse: True. We show that if x ≠ y, then $x^3 \neq y^3$. Let x < y.

- If $0 \leq x < y$, then $x^3 < y^3$, because each of the three factors of y^3 is at least as large as the matching factor of x^3, and y > x.
- If $x < 0 \leq y$, then $x^3 < 0 \leq y^3$.
- If $x < y < 0$, then $0 < -y < -x$, so $(-y)^3 < (-x)^3$, that is, $-y^3 < -x^3$, so $x^3 < y^3$.

In every case $x^3 \neq y^3$. So $x^3 = y^3$ forces x = y.

8. For positive integer n, being divisible by 24 makes n divisible by 4 and by 6. Is the converse also correct?

Solution:

Converse: If both 4 and 6 divide n, then 24 divides n.

Proposition: True. If n = 24k, then n = 4(6k) = 6(4k).

Converse: False. n = 12 is divisible by 4 and by 6, but not by 24. (Since 4 and 6 have the common factor 2, being divisible by both only guarantees divisibility by 12.)

9. For positive integer n, if 60 divides n, both 5 and 12 divide it. Frame and assess the converse.

Solution:

Converse: If both 5 and 12 divide n, then 60 divides n.

Proposition: True. If n = 60k, then n = 5(12k) = 12(5k).

Converse: True. A number divisible by both 5 and 12 is a common multiple of 5 and 12, so it is a multiple of their LCM. Since 5 and 12 have no common factor other than 1, their LCM is 5 × 12 = 60. Hence 60 divides n.

10. For positive integer n, if n equals the square of a prime, then n has exactly three positive divisors. Does the converse hold?

Solution:

Converse: A positive integer with exactly three divisors is the square of a prime.

Proposition: True. Let $n = p^2$ with p prime. A divisor of p^2 is built only from the prime p, used at most twice, so the divisors are 1, p and p^2 — exactly three. (For example, 49 has 1, 7, 49.)

Converse: True. Let n have exactly three divisors: 1, d and n, where 1 < d < n. Since d is a divisor, n/d is also a divisor, so n/d is one of 1, d, n. It cannot be 1 (then d = n), and it cannot be n (then d = 1). So n/d = d, that is, $n = d^2$. If d were not prime, it would have a divisor e with 1 < e < d, and e would also divide $d^2 = n$, giving a fourth divisor of n — a contradiction. So d is prime, and n is the square of a prime.

11. For positive integer n, a product of two different primes has exactly four positive factors. Investigate the converse.

Solution:

Converse: A number with exactly four factors is the product of two different primes.

Proposition: True. For distinct primes p and q, the factors of pq are 1, p, q and pq. (For example, 15 has 1, 3, 5, 15.)

Converse: False. The number 8 has exactly four factors (1, 2, 4, 8), but $8 = 2^3$ is not a product of two different primes.

12. For positive integer n, if n and n + 3 have no common factor apart from 1, then n is not a multiple of 3. Examine the converse.

Solution:

Converse: If n is not a multiple of 3, then n and n + 3 have no common factor apart from 1.

Proposition: True. Suppose n were a multiple of 3, say n = 3k. Then n + 3 = 3(k + 1) is also a multiple of 3, so 3 would be a common factor, which contradicts the given condition. Hence n is not a multiple of 3.

Converse: True. Let d be any common factor of n and n + 3. Then d divides the difference (n + 3) − n = 3, so d = 1 or d = 3. If d = 3, then 3 divides n, which is not the case. So d = 1.

13. Disprove each assertion by finding an n for which the given expression is composite: (i) $4n^2 + 1$ is always prime; (ii) $n^2 + n + 11$ is always prime; (iii) $4^n + 3$ is always prime.

Solution:

(i) Take n = 4: 4(16) + 1 = 65 = 5 × 13. (For n = 1, 2, 3 the values 5, 17, 37 are prime, so n = 4 is the first counterexample.)

(ii) Take n = 10: 100 + 10 + 11 = 121 = 11 × 11. (For n = 1 to 9 the values 13, 17, 23, 31, 41, 53, 67, 83, 101 are all prime, which is why a few checks can never prove the claim.)

(iii) Take n = 4: $4^4 + 3 = 256 + 3 = 259 = 7 \times 37$. (For n = 1, 2, 3 the values 7, 19, 67 are prime.)

Each expression gives a composite value for some n, so none of the three claims is true.

14. Find counterexamples to each statement: (i) whenever n is prime, $2^n - 1$ is prime; (ii) whenever n is even, $2^n + 1$ is prime.

Solution:

(i) For the primes 2, 3, 5, 7 the values 3, 7, 31, 127 are prime. But n = 11 is prime and $2^{11} - 1 = 2047 = 23 \times 89$, which is composite.

(ii) For n = 2 and 4 the values 5 and 17 are prime. But n = 6 is even and $2^6 + 1 = 65 = 5 \times 13$, which is composite.

15. Consider the claim that every multiple of 8 is divisible by both 2 and 4. (i) Prove it. (ii) Would checking divisibility by 2 and 4 alone prove divisibility by 8? Explain.

Solution:

(i) If n = 8k, then n = 2(4k) = 4(2k), so both 2 and 4 divide n.

(ii) No. The number 12 is divisible by 2 and by 4, but not by 8. (Every multiple of 4 is already even, so checking 2 adds nothing.) The correct test for 8 is to check whether the number formed by the last three digits is divisible by 8.

16. Translate the familiar digit-sum divisibility test for 3 into two if–then statements, one for each logical direction.

Solution:

Statement 1: If a whole number is divisible by 3, then the sum of its digits is divisible by 3.

Statement 2: If the sum of the digits of a whole number is divisible by 3, then the number is divisible by 3.

Both are true. Write a number using its digits. For example, $472 = 4 \times 100 + 7 \times 10 + 2 = 4(99 + 1) + 7(9 + 1) + 2 = (4 \times 99 + 7 \times 9) + (4 + 7 + 2)$. The first bracket is a multiple of 9, so it is a multiple of 3. Hence the number equals a multiple of 3 plus its digit sum. The same works for any number of digits, because 10, 100, 1000, … are each 1 more than a multiple of 9. So the number is divisible by 3 exactly when its digit sum is, which gives both statements.

17. We have identified different types of quadrilaterals — squares, rectangles, parallelograms, rhombi, kites and trapezia. One can identify more types (e.g., we could create a category of quadrilaterals that have equal-length opposite sides). Suppose we have identified a category of quadrilaterals called Q, and we have to construct a quadrilateral of this type. For this, we are to use two thin sticks, put them together as diagonals so that the quadrilateral obtained by joining their endpoints is of type Q (see Fig. 9.2).

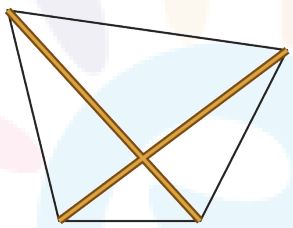


Fig. 9.2

(i) Suppose Q satisfies the following property. If a quadrilateral is of type Q, then it has equal-length diagonals. (a) Should the two sticks be of equal length? Why or why not? (b) Will it matter how the two sticks are put together?

(ii) Instead of the property mentioned above, suppose Q satisfies the following property. If a quadrilateral has equal diagonals, then it is of type Q. What will be your answers to (a) and (b) now?

Solution:

(i) Property: “type Q ⇒ equal diagonals”.

(a) Yes, the two sticks must have the same length. Every quadrilateral of type Q has equal diagonals, so sticks of different lengths could never give a type-Q quadrilateral. Equal diagonals are *necessary*.

(b) It may matter. The property says that type-Q quadrilaterals have equal diagonals. It does not say that every quadrilateral with equal diagonals is of type Q. So, depending on what else Q requires, the way the sticks cross can matter. For example, if Q is “rectangles”, the sticks must cross at their midpoints, and if Q is “squares”, they must cross at their midpoints at right angles.

(ii) Property: “equal diagonals ⇒ type Q”.

(a) Equal sticks will always work, because any quadrilateral with equal diagonals is of type Q. But equal sticks are not forced: this property does not say that all type-Q quadrilaterals have equal diagonals, so Q may also contain quadrilaterals with unequal diagonals.

(b) No. If the two equal sticks cross each other, the quadrilateral formed by joining their ends has equal diagonals, so it is of type Q however the sticks are placed.

Summary: in (i) equal diagonals are necessary, and in (ii) they are sufficient.