Exploring Sequences and Progressions

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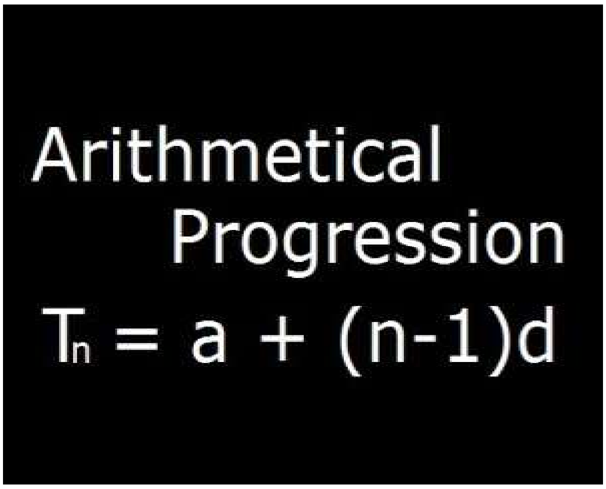
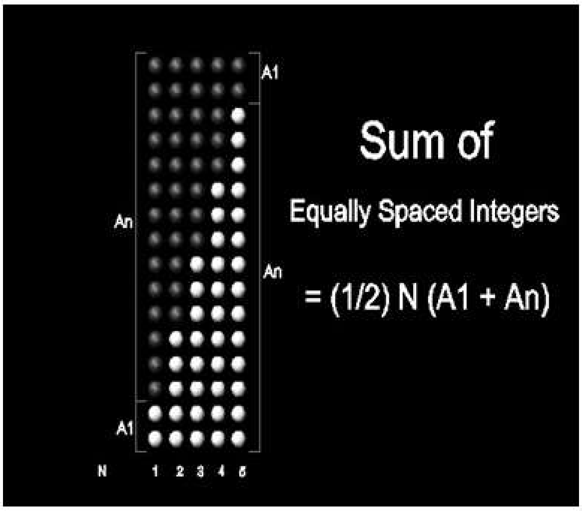
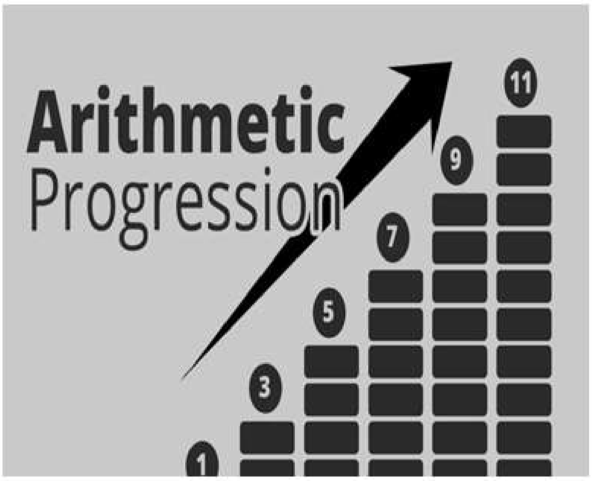






INTRODUCTION

When the terms of a sequence are arranged under a definite rule then they are said to be in a Progression. Progression can be classified mainly into 3 parts as -

(i) Arithmetic Progression (A.P.)

(ii) Geometric Progression (G.P.)

(iii) Harmonic Progression (H.P.)

RECURSIVE RULE FOR A SEQUENCE

So far, we have used a formula in terms of n for the n^{th} term as the explicit rule for a sequence. There is another way of writing the rule for a sequence. Consider the sequence $1, 4, 7, 10, 13, \ldots$

The n^{th} term is $t_n = 3n - 2$ (verify this for yourself). Note that each term is 3 more than its previous term. Thus,

$t_2 = t_1 + 3, t_3 = t_2 + 3,\ t_4 = t_3 + 3$ and so on. In general, we can say $t_n = t_{n-1} + 3$ for $n \geq 2$.

Or we can describe the sequence as $t_1 = 1, t_n = t_{n-1} + 3,$ where n can take the values $2, 3, 4, \ldots$ This way of describing a sequence by relating terms to previous terms is known as a **recursive rule** or **recursive formula**. If you know earlier terms of the sequence, then you can find the next terms using the rule. Note that the earlier terms need to be known to us to use the recursive rule to find the next terms.

Ex.1: Find the first four terms of the sequence given by the recursive rule $u_1 = 1, u_n = 2u_{n-1} + 3$ for $n \geq 2$. Is 133 a term of this sequence?

We successively insert the values of n as $2, 3, 4,$ etc., and compute the values of each term.

$u_2 = 2u_1 + 3 = 2 \times 1 + 3 = 5$

$u_3 = 2u_2 + 3 = 2 \times 5 + 3 = 13$

$u_4 = 2u_3 + 3 = 2 \times 13 + 3 = 29$

Thus, the first four terms of the sequence are 1, 5, 13, 29. Calculating subsequent terms, we can check if 133 is a term of this sequence.

VIRAHĀNKA–FIBONACCI SEQUENCE

A recursive rule or formula does not only have to involve the previous term — it could involve the previous two or more terms.

The most famous example of such a sequence is $V_1, V_2, V_3, \ldots$ where $V_1 = 1, V_2 = 2,$ and

$V_n = V_{n-1} + V_{n-2}$ for $n \geq 3$.

We compute:

$V_3 = V_2 + V_1 = 2 + 1 = 3$

$V_4 = V_3 + V_2 = 3 + 2 = 5$

$V_5 = V_4 + V_3 = 5 + 3 = 8$

So we get the sequence $1, 2, 3, 5, 8, 13, 21, 34, \ldots$

where each term is obtained by adding the previous two.

ARITHMETIC PROGRESSION (A.P.)

Arithmetic Progression is defined as a series in which difference between any two consecutive terms is constant throughout the series. This constant difference is called common difference.

◈ **General form of A.P. :**

If 'a' is the first term and 'd' is the common difference, then an AP can be written as

a , (a + d) , (a + 2d) , (a + 3d) , ...

It is called general form of A.P.

GENERAL TERM OF AN A.P.

General term (n^{th} term) of an AP is given by

$T_n = a + (n-1)\,d$

Note :

(i) General term is also denoted by ℓ (last term)

(ii) n (No. of terms) always belongs to set of natural numbers.

(iii) Common difference can be zero, + ve or – ve.

$d = 0 \Rightarrow$ then all terms of AP are same

Eg. 2, 2, 2, 2, ……… $d = 0$

$d = +ve \Rightarrow$ increasing AP

Eg. $\frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, \ldots\ldots\ldots$ $d = +\frac{1}{2}$

$d = -ve \Rightarrow$ decreasing A.P.

Eg. 57, 52, 47, 42, 37, …….. $d = -\frac{1}{2}$

❖ r^{th} term from end of an A.P.

If number of terms in an A.P. is n then T_r from end $= T_n - (r-1)d = (n - r + 1)^{th}$ from beginning or we can use last term of series as first term and use 'd' with opposite sign of given A.P.

Eg. Consider the expression $t_n = 3n - 7$.

(i) Find its first, second, third, 12th, 18th and 50th terms.

(ii) Which term of the sequence is 332?

(iii) Is 557 a term of this sequence? Why or why not?

❖ EXAMPLES ❖

Ex.2 If the nth term of a sequence be a linear expression in n, then prove that this sequence is an AP.

Sol. Let the nth term of a given progression be given by

$T_n = an + b$, where a and b are constants.

Then, $T_{n-1} = a(n-1) + b = [(an + b) - a]$

$\therefore\ (T_n - T_{n-1}) = (an + b) - [(an + b) - a] = a$,

which is a constant.

Hence, the given progression is an AP.

Ex.3 Write the first three terms in each of the sequences whose n^{th} term is given by -

(i) $a_n = 3n + 2$ (ii) $a_n = n^2 + 1$

Sol. (i) We have,

$a_n = 3n + 2$

Putting n = 1, 2 and 3, we get

$a_1 = 3 \times 1 + 2 = 3 + 2 = 5$,

$a_2 = 3 \times 2 + 2 = 6 + 2 = 8$,

$a_3 = 3 \times 3 + 2 = 9 + 2 = 11$

Thus, the required first three terms of the sequence defined by $a_n = 3n + 2$ are 5, 8, and 11.

(ii) We have,

$a_n = n^2 + 1$

Putting n = 1, 2, and 3 we get

$a_1 = 1^2 + 1 = 1 + 1 = 2$

$a_2 = 2^2 + 1 = 4 + 1 = 5$

$a_3 = 3^2 + 1 = 9 + 1 = 10$

Thus, the first three terms of the sequence defined by $a_n = n^2 + 1$ are 2, 5 and 10.

> **REMARK :** It is evident from the above examples that a sequence is not an A.P. if its nth term is not a linear expression in n.

Ex.4 Which term of the sequence –1, 3, 7, 11, ….. , is 95 ?

Sol. Clearly, the given sequence is an A.P.
We have,
a = first term = –1 and,
d = Common difference = 4.
Let 95 be the n^{th} term of the given A.P. then,
$a_n = 95$
$\Rightarrow a + (n-1)d = 95$
$\Rightarrow -1 + (n-1) \times 4 = 95$
$\Rightarrow -1 + 4n - 4 = 95 \quad \Rightarrow 4n - 5 = 95$
$\Rightarrow 4n = 100 \quad \Rightarrow n = 25$
Thus, 95 is 25^{th} term of the given sequence.

Ex.5 If five times the fifth term of an A.P. is equal to 8 times its eight term, show that its 13^{th} term is zero.

Sol. Let $a_1, a_2, a_3, \ldots, a_n$ be the A.P. with its first term = a and common difference = d.
It is given that $5a_5 = 8a_8$
$\Rightarrow 5(a + 4d) = 8(a + 7d)$
$\Rightarrow 5a + 20d = 8a + 56d \Rightarrow 3a + 36d = 0$
$\Rightarrow 3(a + 12d) = 0 \quad \Rightarrow a + 12d = 0$
$\Rightarrow a + (13-1)d = 0 \quad \Rightarrow a_{13} = 0$

SUM OF n-TERMS OF AN A.P.

Let A.P. be
a, a + d, a + 2d, a + 3d,............., a + (n – 1)d
Then,
$S_n = a + (a + d) + \ldots + \{a + (n-2)d\} + \{a + (n-1)d\}$ ...(i)
$S_n = \{a + (n-1)d\} + \{a + (n-2)d\} + \ldots + (a + d) + a$ ...(ii)
Add (i) & (ii)
$\Rightarrow 2S_n = 2a + (n-1)d + 2a + (n-1)d + \ldots + 2a + (n-1)d$
$\Rightarrow 2S_n = n[2a + (n-1)d] \Rightarrow \mathbf{S_n = \frac{n}{2}[2a + (n-1)d]} \Rightarrow S_n = [a + a + (n-1)d] = \frac{n}{2}[a + \ell]$

$\therefore \mathbf{S_n = \frac{n}{2}[a + \ell]}$ where, ℓ is the last term.

SELECTION OF TERMS IN A.P.

Sometimes we require certain number of terms in A.P. The following ways of selecting terms are generally very convenient.

No. of Terms	Terms	Common Difference
For 3 terms	a – d, a, a + d	d
For 4 terms	a – 3d, a – d, a + d, a + 3d	2d
For 5 terms	a – 2d, a – d, a, a + d, a + 2d	d
For 6 terms	a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d	2d

❖ EXAMPLES ❖

Ex.6 Find the sum of 20 terms of the A.P. 1, 4, 7, 10, ...

Sol. Let a be the first term and d be the common difference of the given A.P. Then, we have a = 1 and d = 3.
We have to find the sum of 20 terms of the given A.P.
Putting a = 1, d = 3, n = 20 in
$S_n = \frac{n}{2}[2a + (n-1)d]$, we get
$S_{20} = \frac{20}{2}[2 \times 1 + (20-1) \times 3] = 10 \times 59 = 590$

Ex.7 Find the sum of first 30 terms of an A.P. whose second term is 2 and seventh term is 22.

Sol. Let a be the first term and d be the common difference of the given A.P. Then,

$a_2 = 2$ and $a_7 = 22$

$\Rightarrow$ $a + d = 2$ and $a + 6d = 22$

Solving these two equations, we get

$a = -2$ and $d = 4$.

$S_n = \frac{n}{2}[2a + (n-1)d]$

$\therefore$ $S_{30} = \frac{30}{2}[2 \times (-2) + (30-1) \times 4]$

$\Rightarrow$ $15(-4 + 116) = 15 \times 112 = 1680$

Hence, the sum of first 30 terms is 1680.

Ex.8 The sum of three numbers in A.P. is –3, and their product is 8. Find the numbers.

Sol. Let the numbers be $(a - d), a, (a + d)$. Then,

Sum $= -3 \Rightarrow (a - d) + a + (a + d) = -3$

$\Rightarrow$ $3a = -3$

$\Rightarrow a = -1$

Product = 8

$\Rightarrow$ $(a - d)(a)(a + d) = 8$

$\Rightarrow$ $a(a^2 - d^2) = 8$

$\Rightarrow$ $(-1)(1 - d^2) = 8$

$\Rightarrow$ $d^2 = 9$ $\Rightarrow$ $d = \pm 3$

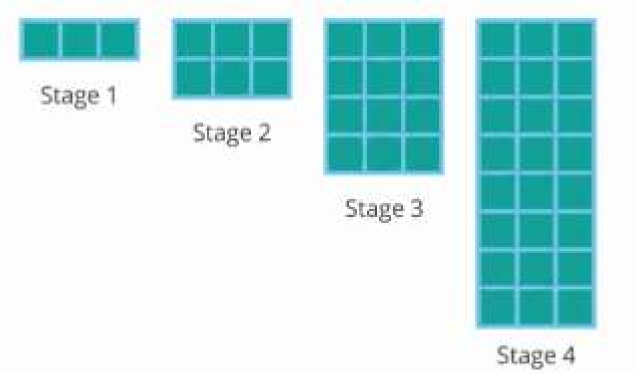
If $d = 3$, the numbers are –4, –1, 2.

If $d = -3$, the numbers are 2, – 1, –4.

Thus, the numbers are –4, –1, 2, or 2, – 1, – 4.

GEOMETRIC PROGRESSIONS (GP)

Consider the growing pattern of squares shown in Fig.. The first four stages of the pattern are shown



If we count the total number of green squares in the four stages of the pattern, we get the sequence 3, 6, 12, 24. These may be rewritten as 3, 3 × 2, 3 × 4, 3 × 8, … or as 3, 3 × 2, 3 × 22 , 3 × 23 , … If we treat these expressions as the terms of a sequence, we get $t_1 = 3$, $t_2 = 3 \times 2$, $t_3 = 3 \times 2^2$, $t_4 = 3 \times 2^3$ and so on.

We can write the n^{th} term as $t_n = 3 \times 2^{n-1}$

As a recursive formula, we can write $t_1 = 3$ and $t_n = 2t_{n-1}$ for $n \geq 2$.

The first six terms of the sequence are 3, 6, 12, 24, 48, 96. Each term of the sequence is obtained by multiplying the previous term by 2. This constant multiplier is also known as the common ratio of the sequence. We see that the ratios of consecutive pairs of terms are all the same.

Such a sequence with a common ratio is known as a **Geometric Progression or GP**. In general, a GP may be described in the form of a sequence as $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$,

where ‘ a ’ is the **first term**, ‘ r ’ is the **common ratio** and $t_n = ar^{n-1}$ is the n^{th} term.

Ex.9 Is $1, 2, 4, 8, 16, \ldots$ a geometric progression? If so, what is the common ratio?

Ans. **Yes, the sequence (1, 2, 4, 8, 16, …..) is a geometric progression**, and its **common ratio is (2)**.

Ex. 10 Is 1, 3, 9, 27, 81, … a geometric progression? If so, what is the common ratio?

Ans. **Yes, the sequence (1, 3, 9, 27, 81, ….) is a geometric progression**, and its common ratio is **3**

Ex.11 Check whether the following sequences are geometric progressions and find their n^{th} terms.

(i) $2, 10, 50, 250, \ldots$ (ii) $4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \ldots$ (iii) $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$

Ex.12 Find the ninth term of the G.P. 1, 4, 16, 64, …

Sol. The ninth term of the G.P. 1, 4, 16, 64, ….

We know that,

$t_1 = a = 1$, $r = t_2/t_1 = 4/1 = 4$

By using the formula,

$T_n = ar^{n-1}$

$T_9 = 1\ (4)^{9-1} = 1\ (4)^8 = 4^8$

Ex.13 Which term of the progression 0.004, 0.02, 0.1, …. is 12.5?

Sol. By using the formula,

$T_n = ar^{n-1}$

Given:

$a = 0.004$

$r = t_2/t_1 = (0.02/0.004)$

$= 5$

$T_n = 12.5$

$n = ?$

So, $T_n = ar^{n-1}$

$12.5 = (0.004)\ (5)^{n-1}$

$12.5/0.004 = 5^{n-1}$

$3000 = 5^{n-1}$

$5^5 = 5^{n-1}$

$5 = n-1$

$n = 5 + 1 = 6$

Depending on the common ratio (r), choose the formula that avoids a negative denominator

For $r > 1: S_n = \dfrac{a\left(r^n - 1\right)}{r - 1}$

For $r < 1: \ S_n = \dfrac{a\left(1 - r^n\right)}{1 - r}$

For $r = 1$: $S_n = na$

Ex.14 Find the sum of the first 6 terms of the G.P. $4, 12, 36, \ldots$

Sol. First term, $a = 4$

Common ratio, $r = \dfrac{12}{4} = 3$

$n = 6$

Sum of n terms of a G.P.,

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$S_6 = \frac{4\left(3^6 - 1\right)}{3 - 1} = \frac{4(729 - 1)}{2} = 2 \times 728 = 1456$$