Exploring algebraic identities

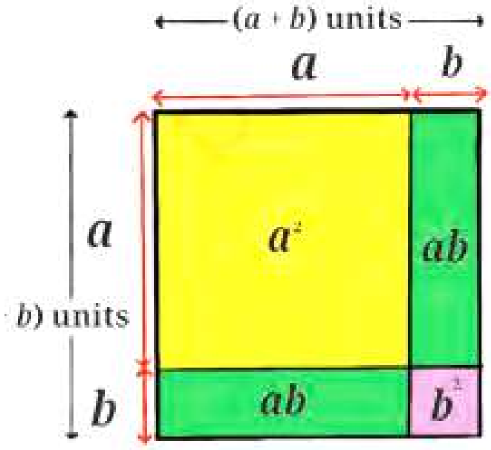
WHAT IS AN ALGEBRAIC EXPRESSION?

An algebraic expression in mathematics is an expression which is made up of variables and constants, along with algebraic operations (addition, subtraction, etc.). Expressions are made up of terms.

Examples : $3x + 4y - 7$, $4x - 10$, etc.

VISUALISING IDENTITIES

Identity-1 $(a + b)^2 = a^2 + 2ab + b^2$



Square of side (a + b) units

The area of the outer square is $(a + b)^2$. The area of the larger square inside the outer square is a^2 while the area of the smaller square is b^2 . The areas of the two rectangles are ab each. Together they make the bigger square; hence we can conclude that $(a + b)^2 = a^2 + 2ab + b^2$

Ex.1 Let $a = -2$ and $b = -3$.
Then $(a+b) = -5$ and $(a+b)^2 = 25$.

Also $a^2 = 4, b^2 = 9$ and $2ab = 12$.

Thus $a^2 + 2ab + b^2 = 4 + 12 + 9 = 25$.

Ex.2 Let us try to expand $(5x+2y)^2$. Here $a = 5x$ and $b = 2y$.

Then $(5x+2y)^2 = (5x)^2 + 2(5x)(2y) + (2y)^2 = 25x^2 + 20xy + 4y^2$.

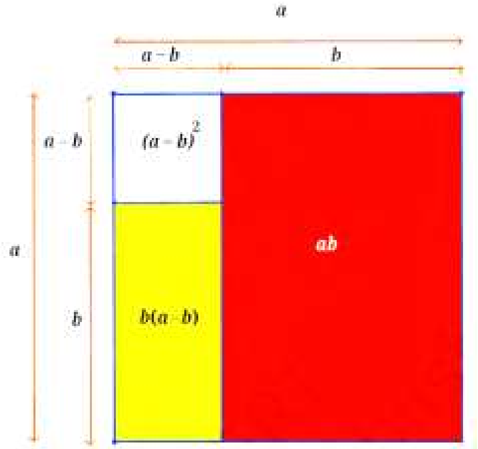
We can also use the identity to help us with numerical calculations.

Ex.3 To calculate 43^2, we can write it as

$(40+3)^2 = 40^2 + 2\times 40\times 3 + 3^2 = 1600 + 240 + 9 = 1849$.

Identity-2 $\quad$ **$(a - b)^2 = a^2 - 2ab + b^2$**

$(a - b)^2 = a^2 - ab - b(a - b) = a^2 - ab - ba + b^2 = a^2 - 2ab + b^2$.



Ex.4 Suppose we have to calculate 29^2 . We can express this a

$$(30-1)^2 = 30^2 - 2\times 30\times 1 + 1^2 = 900 - 60 + 1 = 841$$

Ex.5 Expand each of the following :

(i) $(3x - 4y)^2$ $\quad$ (ii) $\left(\frac{x}{2}+\frac{y}{3}\right)^2$

Sol. (i) We have,

$(3x - 4y)^2 = (3x)^2 - 2 \times 3x \times 4y + (4y)^2 = 9x^2 - 24xy + 16y^2$

(ii) We have,

$$\left(\frac{x}{2}+\frac{y}{3}\right)^2 = \left(\frac{x}{2}\right)^2 + 2\times\frac{x}{2}\times\frac{y}{3} + \left(\frac{y}{3}\right)^2 = \frac{x^2}{4}+\frac{1}{3}xy+\frac{y^2}{9}$$

Identity-3 $\quad$ **$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$**

Ex.6 Write the following in expanded form $(9x + 2y + z)^2$

Sol. Using the identity

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

We have, $(9x + 2y + z)^2$

$= (9x)^2 + (2y)^2 + z^2 + 2 \times 9x \times 2y + 2 \times 2y \times z + 2 \times 9x \times z$

$= 81x^2 + 4y^2 + z^2 + 36xy + 4yz + 18xz$

Ex.7 Let us use this identity to find the square of a number, say 119 :

$$119^2\ (100+10+9)^2 = 100^2 + 10^2 + 9^2 + 2(100)(10) + 2(100)(9) + 2(10)(9)$$

$$= 10000 + 100 + 81 + 2000 + 1800 + 180 = 14161$$

Identity-4 $\quad$ **$a^2 - b^2 = (a + b)(a - b)$**

Ex.8 Find the products :

(i) $(2x + 3y)(2x - 3y)$ $\quad$ (ii) $\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)\left(x^4+\frac{1}{x^4}\right)$

Sol. (i) We have,

$(2x + 3y)(2x - 3y) = (2x)^2 - (3y)^2$

[Using: $(a + b)(a - b) = a^2 - b^2$]$= (2x)^2 - (3y)^2 = 4x^2 - 9y^2$

(ii) We have,

$$\left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)\left(x^4+\frac{1}{x^4}\right) = \left(x^2-\frac{1}{x^2}\right)\left(x^2+\frac{1}{x^2}\right)\left(x^4+\frac{1}{x^4}\right)$$

$$= \left\{(x^2)^2 - \left(\frac{1}{x^2}\right)^2\right\}\left(x^4+\frac{1}{x^4}\right) = \left(x^4-\frac{1}{x^4}\right)\left(x^4+\frac{1}{x^4}\right) = (x^4)^2 - \left(\frac{1}{x^4}\right)^2 = x^8 - \frac{1}{x^8}$$

Ex.9 Evaluate each of the following by using identities :

(i) 103×97 (ii) 103×103 (iii) $(97)^2$ (iv) $185 \times 185 - 115 \times 115$

Sol. (i) We have,

$103 \times 97 = (100 + 3)(100 - 3) = (100)^2 - (3)^2 = 10000 - 9 = 9991$

(ii) We have,

$103 \times 103 = (103)^2 = (100 + 3)^2 = (100)^2 + 2 \times 100 \times 3 + (3)^2 = 10000 + 600 + 9 = 10609$

(iii) We have,

$(97)^2 = (100 - 3)^2 = (100)^2 - 2 \times 100 \times 3 + (3)^2 = 10000 - 600 + 9 = 9409$

(iv) We have,

$185 \times 185 - 115 \times 115 = (185)^2 - (115)^2 = (185 + 115)(185 - 115) = 300 \times 70 = 21000$

Identity-5 $\mathbf{(x + a)(x + b) = x^2 + (a + b)x + ab}$

Identity-6 $\mathbf{(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd}$

Identity-7 $\mathbf{(a + b)^3 = a^3 + b^3 + 3ab(a + b)}$

Identity-8 $\mathbf{(a - b)^3 = a^3 - b^3 - 3ab(a - b)}$

Ex.10 Write each of the following in expanded form:

(i) $(2x + 3y)^3$ (ii) $(3x - 2y)^3$

Sol. (i) Replacing a by 2x and b by 3y in the identity

$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$, we have

$(2x + 3y)^3 = (2x)^3 + (3y)^3 + 3 \times 2x \times 3y \times (2x + 3y)$

$= 8x^3 + 27^3 + 18xy \times 2x + 18xy \times 3y$

$= 8x^3 + 27y^3 + 36x^2y + 54xy^2$

(ii) Replacing a by 3x and b by 2y in the identity $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$, we have

$(3x - 2y)^3 = (3x)^3 - (2y)^3 - 3 \times 3x \times 2y \times (3x - 2y)$

$= 27x^3 - 8y^3 - 18xy \times (3x - 2y)$

$= 27x^3 - 8y^3 - 54x^2y + 36xy^2$

Ex.11 If $x + y = 12$ and $xy = 27$, find the value of $x^3 + y^3$.

Sol. We know that

$(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

Putting $x + y = 12$ and $xy = 27$ in the above identity, we get

$12^3 = x^3 + y^3 + 3 \times 27 \times 12$

$\Rightarrow 1728 = x^3 + y^3 + 972$

$\Rightarrow x^3 + y^3 = 1728 - 972$

$\Rightarrow x^3 + y^3 = 756$

Identity-9 $\mathbf{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$

Identity-10 $\mathbf{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$

Ex.12 Simplify each of the following:

(i) $(x + 3)^3 + (x - 3)^3$

Here $a = (x + 3)$, $b = (x - 3)$

$$= (x+3+x-3)\left[(x+3)^2 + (x-3)^2 - (x+3)(x-3)\right]$$

$$= 2x\left[(x^2+9+6x) + (x^2+9-6x) - x^2 + 9\right] = 2x\left[(x^2+9+6x+x^2+9-6x-x^2+9)\right]$$

$$= 2x(x^2+27) = 2x^3 + 54x$$

(ii) $(x/2 + y/3)^3 - (x/2 - y/3)^3$

Here $a = (x/2 + y/3)$ and $b = (x/2 - y/3)$

$$=\left[\left(\frac{x}{2}+\frac{y}{3}\right)-\left(\frac{x}{2}-\frac{y}{3}\right)\right]\left[\left(\frac{x}{2}+\frac{y}{3}\right)^2+\left(\frac{x}{2}-\frac{y}{3}\right)^2+\left(\frac{x}{2}+\frac{y}{3}\right)\left(\frac{x}{2}-\frac{y}{3}\right)\right]$$

$$=\frac{2y}{3}\left[\left(\frac{x^2}{4}+\frac{y^2}{9}+\frac{2xy}{6}\right)+\left(\frac{x^2}{4}+\frac{y^2}{9}-\frac{2xy}{6}\right)+\frac{x^2}{4}-\frac{y^2}{9}\right]$$

$$=\frac{2y}{3}\left[\frac{x^2}{4}+\frac{y^2}{9}+\frac{x^2}{4}+\frac{x^2}{4}\right]=\frac{2y}{3}\left[\frac{3x^2}{4}+\frac{y^2}{9}\right]=\frac{x^2y}{2}+\frac{2y^3}{27}$$

Identity-11 $\quad$ **$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$**

Special case : if $a + b + c = 0$ then $a^3 + b^3 + c^3 = 3abc$.

Ex.13 If $x + y + z = 1$, $xy + yz + zx = -1$ and $xyz = -1$, find the value of $x^3 + y^3 + z^3$.

Sol. We know that :

$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

$\Rightarrow x^3 + y^3 + z^3 - 3xyz$

$= (x + y + z)(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx - 3xy - 3yz - 3zx)$

[Adding and subtracting $2xy + 2yz + 2zx$]

$\Rightarrow x^3 + y^3 + z^3 - 3xyz = (x + y + z)\{(x + y + z)^2 - 3(xy + yz + zx)\}$

$\Rightarrow x^3 + y^3 + z^3 - 3 \times -1 = 1 \times \{(1)^2 - 3 \times -1\}$

[Putting the values of $x + y + z$, $xy + yz + zx$, and xyz]

$\Rightarrow x^3 + y^3 + z^3 + 3 = 4 \Rightarrow x^3 + y^3 + z^3 = 4 - 3 \Rightarrow x^3 + y^3 + z^3 = 1$

FACTORIZATION

Type I : Factorization by taking out the common factors.

Ex.14 Factorize the following expression : $2x^2y + 6xy^2 + 10x^2y^2$

Sol. $2x^2y + 6xy^2 + 10x^2y^2 = 2xy(x + 3y + 5xy)$

Type II : Factorization by grouping the terms.

Ex.15 Factorize the following expression :

$a^2 - b + ab - a$

Sol. $a^2 - b + ab - a = a^2 + ab - b - a = (a^2 + ab) - (b + a)$

$= a(a + b) - (a + b) = (a + b)(a - 1)$

Type III : Factorization by making a perfect square.

Ex.16 Factorize the following expression :

$9x^2 + 12xy + 4y^2$

Sol. $9x^2 + 12xy + 4y^2 = (3x)^2 + 2 \times (3x) \times (2y) + (2y)^2 = (3x + 2y)^2$

Type IV : Factorizing by difference of two squares.

Ex.17 Factorize each of the following expressions :

(i) $36x^2 - 12x + 1 - 25y^2$ $\qquad$ (ii) $a^2 - \dfrac{9}{a^2}$, $a \neq 0$

Sol. (i) $36x^2 - 12x + 1 - 25y^2 = (6x)^2 - 2 \times 6x \times 1 + 1^2 - (5y)^2 = (6x-1)^2 - (5y)^2$

$= \{(6x - 1) - 5y\}\{(6x - 1) + 5y\} = (6x - 1 - 5y)(6x - 1 + 5y) = (6x - 5y - 1)(6x + 5y - 1)$

(ii) $a^2 - \dfrac{9}{a^2} = (a)^2 - \left(\dfrac{3}{a}\right)^2 = \left(a - \dfrac{3}{a}\right)\left(a + \dfrac{3}{a}\right)$

Type V : Factorizing the sum and difference of cubes of two quantities.

(i) $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$ (ii) $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$

Ex.18 Factorize the following expression : $a^3 + 27$

Sol. $a^3 + 27 = a^3 + 3^3 = (a + 3)(a^2 - 3a + 9)$

◈ Factorization of quadratic polynomial by splitting the middle team

Type I : Factorization of Quadratic polynomials of the form $x^2 + bx + c$.

Ex.19 Factorize the following quadratic polynomials: $x^2 - 21x + 108$

Sol. In order to factorize $x^2 - 21x + 108$, we have to find two numbers such that their sum is -21 and the product 108.

Clearly, $-21 = -12 - 9$ and $-12 \times -9 = 108$

$\therefore\ x^2 - 21x + 108 = x^2 - 12x - 9x + 108$

$= (x^2 - 12x) - (9x - 108)$

$= x(x - 12) - 9(x - 12) = (x - 12)(x - 9)$

Ex.20 Factorize the following by splitting the middle term : $x^2 + 3\sqrt{3}x + 6$

Sol. In order to factorize $x^2 + 3\sqrt{3}x + 6$, we have to find two numbers p and q such that

$p + q = 3\sqrt{3}$ and $pq = 6$

Clearly, $2\sqrt{3} + \sqrt{3} = 3\sqrt{3}$ and $2\sqrt{3} \times \sqrt{3} = 6$

So, we write the middle term $3\sqrt{3}x$ as $2\sqrt{3}x + \sqrt{3}x$, so that

$x^2 + 3\sqrt{3}x + 6 = x^2 + 2\sqrt{3}x + \sqrt{3}x + 6$

$= (x^2 + 2\sqrt{3}x) + (\sqrt{3}x + 6) = (x^2 + 2\sqrt{3}x) + (\sqrt{3}x + 2\sqrt{3} \times \sqrt{3})$

$= x(x + 2\sqrt{3}) + \sqrt{3}(x + 2\sqrt{3}) = (x + 2\sqrt{3})(x + \sqrt{3})$

Type II : Factorization of polynomials reducible to the form $x^2 + bx + c$.

Ex.21 Factorize : $(a^2 - 2a)^2 - 23(a^2 - 2a) + 120$.

Sol. Let $a^2 - 2a = x$. Then,

$(a^2 - 2a)^2 - 23(a^2 - 2a) + 120 = x^2 - 23x + 120$

Now, $x^2 - 23x + 120 = x^2 - 15x - 8x + 120$

$= (x^2 - 15x) - (8x - 120) = x(x - 15) - 8(x - 15) = (x - 15)(x - 8)$

Replacing x by $a^2 - 2a$ on both sides, we get

$(a^2 - 2a)^2 - 23(a^2 - 2a) + 120 = (a^2 - 2a - 15)(a^2 - 2a - 8)$

$= (a^2 - 5a + 3a - 15)(a^2 - 4a + 2a - 8) = \{(a(a - 5) + 3(a - 5)\}\{a(a - 4) + 2(a - 4)\}$

$= \{(a - 5)(a + 3)\}\{(a - 4)(a + 2)\} = (a - 5)(a + 3)(a - 4)(a + 2)$

Type III : Factorization of quadratic polynomials of the form $ax^2 + bx + c$, $a \neq 0, 1$

Ex.22 Factorize the following expression : $6x^2 - 5x - 6$

Sol. The given expression is of the form $ax^2 + bx + c$, where, $a = 6$, $b = -5$ and $c = -6$.

In order to factorize the given expression, we have to find two numbers ℓ and m such that

$\ell + m = b =$ i.e, $\ell + m = -5$

and $\ell m = ac$ i.e. $\ell m = 6 \times -6 = -36$

i.e., we have to find two factors of -36 such that their sum is -5. Clearly,

$-9 + 4 = -5$ and $-9 \times 4 = -36$

$\therefore\ \ell = -9$ and $m = 4$

Now, we split the middle term $-5x$ of $6x^2 - 5x - 6$ as $-9x + 4x$, so that

$6x^2 - 5x - 6 = 6x^2 - 9x + 4x - 6$

$= (6x^2 - 9x) + (4x - 6) = 3x(2x - 3) + 2(2x - 3) = (2x - 3)(3x + 2)$

Ex.23 Factorize : $\sqrt{3}\,x^2 + 11x + 6\sqrt{3}$

Sol. The given quadratic expression is of the form $ax^2 + bx + c$, where $a = \sqrt{3}$, $b = 11$ and $c = 6\sqrt{3}$.

In order to factorize it, we have to find two numbers ℓ and m such that $\ell + m = b = 11$ and $\ell m = ac = \sqrt{3} \times 6\sqrt{3} = 18$

Clearly, $9 + 2 = 11$ and $9 \times 2 = 18$

$\therefore\ \ell = 9$ and $m = 2$

Now, $\sqrt{3}\,x^2 + 11\,x + 6\sqrt{3} = \sqrt{3}\,x^2 + 9x + 2x + 6\sqrt{3} \quad = (\sqrt{3}\,x^2 + 9x) + (2x + 6\sqrt{3})$

$= (\sqrt{3}\,x^2 + 3\sqrt{3} \times \sqrt{3}\,x) + (2x + 6\sqrt{3}) = \sqrt{3}\,x\,(x + 3\sqrt{3}) + 2(x + 3\sqrt{3})$

$= (\sqrt{3}\,x + 2)\,(x + 3\sqrt{3})$.

Hence, $\sqrt{3}\,x^2 + 11\,x + 6\sqrt{3} = (\sqrt{3}\,x + 2)\,(x + 3\sqrt{3})$

❖ Application of Factor theorem in the factorization of polynomials

(i) Obtain the polynomial p(x)

(ii) Obtain the constant term in p(x) and find its all possible factors. For example, in the polynomial $x^4 + x^3 - 7x^2 - x + 6$ the constant term is 6 and its factors are $\pm 1, \pm 2, \pm 3, \pm 6$.

(iii) Take one of the factors, say a and replace x by it in the given polynomial. If the polynomial reduces to zero, then $(x - a)$ is a factor of polynomial.

(iv) Obtain the factors equal in no. to the degree of polynomial. Let these are $(x-a)$, $(x-b)$, $(x-c.)$.....

(v) Write $p(x) = k\,(x-a)\,(x-b)\,(x-c)$ ..... where k is constant.

(vi) Substitute any value of x other than a,b,c ...... and find the value of k.

Ex.24 Using factor theorem, factorize the polynomial $x^3 - 6x^2 + 11\,x - 6$.

Sol. Let $f(x) = x^3 - 6x^2 + 11x - 6$

The constant term in f(x) is equal to -6 and factors of -6 are $\pm 1, \pm 2, \pm 3, \pm 6$.

Putting $x = 1$ in f(x), we have

$f(1) \quad = 1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 1 - 6 + 11 - 6 = 0$

$\therefore\ (x - 1)$ is a factor of f(x)

Similarly, $x - 2$ and $x - 3$ are factors of f(x).

Since f(x) is a polynomial of degree 3. So, it can not have more than three linear factors.

Let $f(x) = k\,(x-1)\,(x-2)\,(x-3)$. Then,

$x^3 - 6x^2 + 11x - 6 = k(x-1)\,(x-2)\,(x-3)$

Putting $x = 0$ on both sides, we get

$-6 = k\,(0-1)\,(0-2)\,(0-3) \Rightarrow -6 = -6\,k \Rightarrow k = 1$

Putting $k = 1$ in $f(x) = k\,(x-1)\,(x-2)\,(x-3)$, we get

$f(x) = (x-1)\,(x-2)\,(x-3)$

Hence, $x^3 - 6x^2 + 11x - 6 = (x-1)\,(x-2)\,(x-3)$

Ex.25 Factorize, $2x^4 + x^3 - 14x^2 - 19x - 6$

Sol. Let $f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$ be the given polynomial. The factors of the constant term -6 are $\pm 1, \pm 2, \pm 3$ and ± 6, we have,

$f(-1) = 2(-1)^4 + (-1)^3 - 14(-1)^2 - 19(-1) - 6 = 2 - 1 - 14 + 19 - 6 = 21 - 21 = 0$

and, $f(-2) = 2(-2)^4 + (-2)^3 - 14(-2)^2 - 19(-2) - 6 = 32 - 8 - 56 + 38 - 6 = 0$

So, $x + 1$ and $x + 2$ are factors of f(x).

$\Rightarrow\ (x + 1)\,(x + 2)$ is also a factor of f(x) $\Rightarrow x^2 + 3x + 2$ is a factor of f(x)

Now, we divide

$f(x) = 2x^4 + x^3 - 14x^2 - 19x - 6$ by

$x^2 + 3x + 2$ to get the other factors.

$$\begin{array}{r|l} & \quad 2x^2 - 5x - 3 \\ \hline x^2+3x+2 & 2x^4 + x^3 - 14x^2 - 19x - 6 \\ & 2x^4 + 6x^3 + 4x^2 \\ & - \quad - \quad - \\ \hline & \quad -5x^3 - 18x^2 - 19x - 6 \\ & \quad -5x^3 - 15x^2 - 10x \\ & \quad + \quad + \quad + \\ \hline & \qquad -3x^2 - 9x - 6 \\ & \qquad -3x^2 - 9x - 6 \\ & \qquad + \quad + \quad + \\ \hline & \qquad 0 \end{array}$$

$\therefore\ 2x^4 + x^3 - 14x^2 - 19x - 6 = (x^2 + 3x + 2)(2x^2 - 5x - 3) = (x + 1)(x + 2)(2x^2 - 5x - 3)$

Now $2x^2 - 5x - 3 \quad = 2x^2 - 6x + x - 3 = 2x(x - 3) + 1(x - 3) = (x - 3)(2x + 1)$

Hence, $2x^4 + x^3 - 14x^2 - 19x - 6 = (x + 1)(x + 2)(x - 3)(2x+1)$

WORKING RULE TO DIVIDE A POLYNOMIAL BY ANOTHER POLYNOMIAL

Step 1: First arrange the term of dividend and the divisor in the decreasing order of their degrees.

Step 2 : To obtain the first term of quotient divide the highest degree term of the dividend by the highest degree term of the divisor.

Step 3: To obtain the second term of the quotient, divide the highest degree term of the new dividend obtained as remainder by the highest degree term of the divisor.

Step 4 : Continue this process till the degree of remainder is less than the degree of divisor.

◈ Division Algorithm for Polynomial

If p(x) and g(x) are any two polynomials with

$g(x) \neq 0$, then we can find polynomials q(x) and r(x) such that

$$p(x) = q(x) \times g(x) + r(x)$$

where $r(x) = 0$ or degree of r(x) < degree of g(x).

The result is called Division Algorithm for polynomials.

$$\boxed{\textbf{Dividend} = \textbf{Quotient} \times \textbf{Divisor} + \textbf{Remainder}}$$

Ex.26 Divide $3x^3 + 16x^2 + 21x + 20$ by $x + 4$.

Sol.

$$\begin{array}{r|l} & 3x^2 + 4x + 5 \\ \hline x+4 & 3x^3 + 16x^2 + 21x + 20 \\ & 3x^3 + 12x^2 \\ & - \quad - \\ \hline & \qquad 4x^2 + 21x + 20 \\ & \qquad 4x^2 + 16x \\ & \qquad - \quad - \\ \hline & \qquad\qquad 5x + 20 \\ & \qquad\qquad 5x + 20 \\ & \qquad\qquad - \quad - \\ \hline & \qquad\qquad 0 \end{array}$$

First term of $q(x) = \frac{3x^3}{x} = 3x^2$

Second term of $q(x) = \frac{4x^2}{x} = 4x$

Third term of $q(x) = \frac{5x}{x} = 5$

Quotient $= 3x^2 + 4x + 5$ Remainder $= 0$