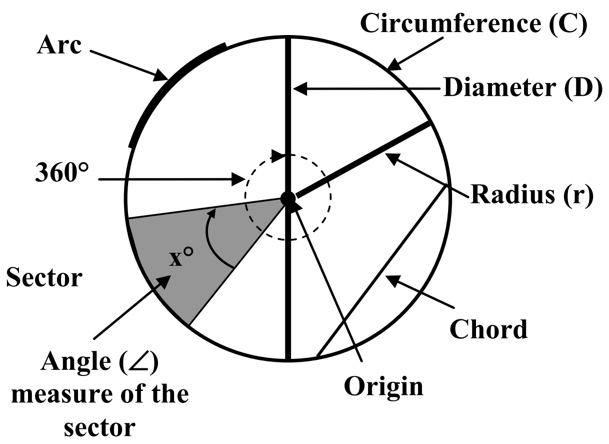
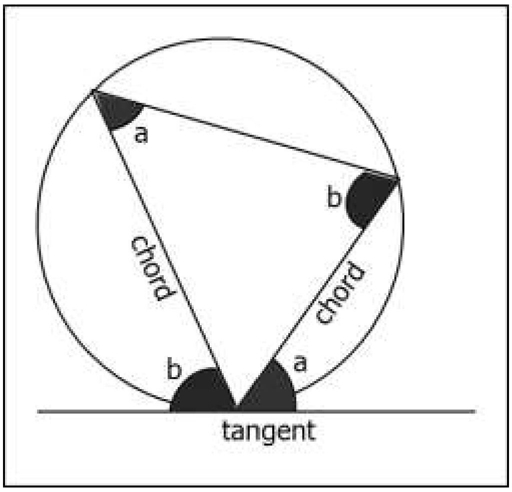
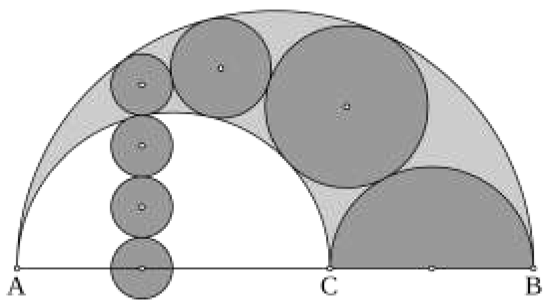
I’m Up and Down, and Round and Round

Chapter Outline

- ✧ **Terms & Definitions**
- ✧ **Theorems related to circle**
- ✧ **Cyclic Quadrilaterals**







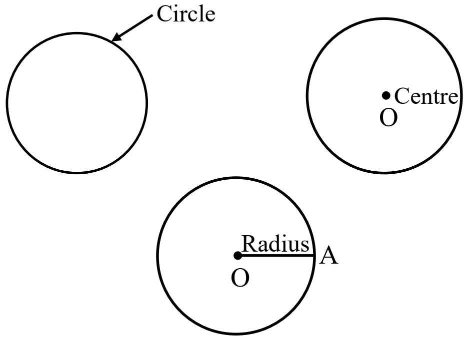
TERMS & DEFINITIONS

1. **Circle :** A circle is a collection of all those points in a plane that are at a given constant distance from a given fixed point in the plane.

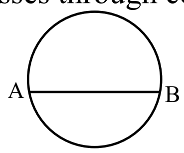
2. **Centre :** The fixed point is called the centre of the circle. In the figure O is the centre.

Everyday Examples
Think of the full moon, the sun during a solar eclipse, or ripples formed when a raindrop hits a puddle. All of these represent perfect circles in nature.

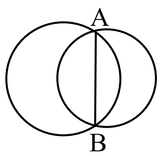
3. **Radius :** The constant distance from its centre is called the radius of the circle. In the figure, OA is radius.



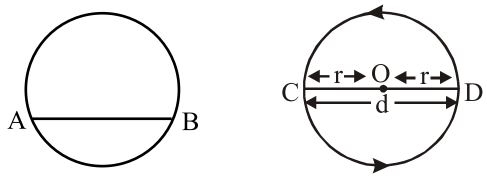
4. **Chord :** A line segment joining two points on a circle is called a chord of the circle. In the figure, AB is a chord of the circle. If a chord passes through centre then it is longest chord.



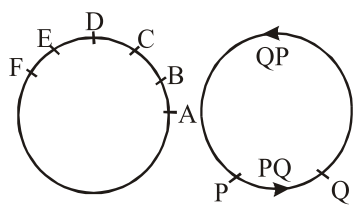
5. **Common Chord :** A line joining common points of two intersecting circles is called common chord. AB is common chord.



6. **Diameter :** A chord passing through the centre of a circle is called the diameter of the circle. A circle has an infinite number of diameters. CD is the diameter of the circle as shown in the figure. If d is the diameter of the circle then d = 2r. where r is the radius. or the longest chord is called diameter.
In the figure, CD is the diameter and the arcs $\overset{\frown}{CD}$ and $\overset{\frown}{DC}$ are semicircles.

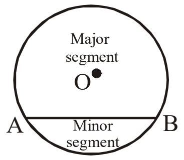


7. **Arc :** A continuous piece of a circle is called an arc. Let A,B,C,D,E,F be the points on the circle. The circle is divided into different pieces. Then, the pieces AB, BC, CD, DE, EF etc. are all arcs of the circle.

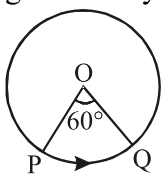


8. **Circumference of a circle :** The perimeter of a circle is called its circumference. The circumference of a circle of radius r is $2\pi r$.

9. Segment : Let AB be a chord of the circle. Then, AB divides the region enclosed by the circle (i.e., the circular disc) into two parts. Each of the parts is called a segment of the circle. The segment, containing the minor arc is called minor segment and the segment, containing the major arc, is called the major segment and segment of a circle is the region between an arc and a chord of the circle.



10. Central Angles : Consider a circle. The angle subtended by an arc at the centre O is called the central angle. The vertex of the central angle is always at the centre O.



11. Degree measure of an arc : Degree measure of a minor arc is the measure of the central angle subtended by the arc.

In the figure, the measure of the arc $\overset{\frown}{PQ}$ is 60° i.e., $m\overset{\frown}{PQ} = 60°$. The measure of a major arc is $360° - m\overset{\frown}{PQ}$.

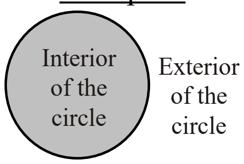
The degree measure of the major arc is $360° - 60° = 300°$

$\therefore\ m\overset{\frown}{QP} = 300°$.

The degree measure of the circumference of the circle is always 360°.

12. Interior and Exterior of Circle :

A circle divides the plane on which it lies into three parts.



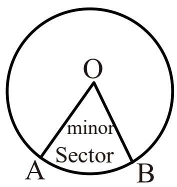
(i) Inside the circle, which is called the interior of the circle

(ii) Circumference of circle.

(iii) Outside the circle, which is called the exterior of the circle.

The circle and its interior make up the circular region.

13. Sector :

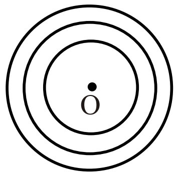


A sector is that region of a circular disc which lies between an arc and the two radii joining the extremities of the arc and the centre. OAB is a sector as shown in the figure.

14. Quadrant. One fourth of a circular disc is called a quadrant.

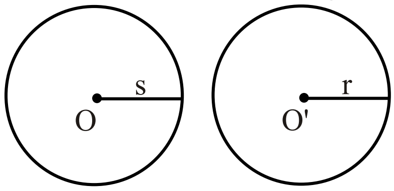
15. Concentric Circles :

Circles having the same centre and different radius are said to be concentric circles.



Concentric Circles

16. Congruent circles : Two circles are said to be congruent if and only if, one of them can be superimposed on the other, so as to cover it exactly. It means two circles are congruent if and only if, their radii are equal. i.e., C (O, r) and C (O′ , s) are congruent if and only if r = s.



Congruent circles if r = s

17. Congruent arcs : Two arcs of a circle are congruent, if either of them can be superimposed on the other, so as to cover it exactly. It is only possible, if degree measure of two arcs are the same.

SYMMETRIES AND CIRCLES THROUGH POINTS

Circles are incredibly appealing because they are perfectly symmetrical.

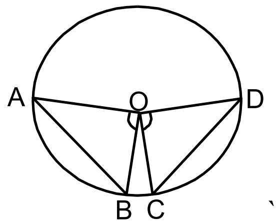
- Rotational Symmetry: If you rotate a circle by any angle around its centre, it looks exactly the same.
- Reflection Symmetry: Every diameter acts as a line of symmetry. If you fold a circular paper along its diameter, the boundaries will overlap perfectly.
- Circles through 1 or 2 points: You can draw an infinite number of circles passing through two points (A and B). The centres of all these circles will lie on the perpendicular bisector of the line segment AB.
- Circles through 3 points: There is one, and only one, unique circle that can pass through three non-collinear points. This is called a circumcircle, and its centre is the circumcentre.

CHORDS AND THE ANGLES THEY SUBTEND

Theorem : Equal chords of a circle subtend equal angles at the centre.

Given : AB and CD are the two equal chords of a circle with centre O.

To Prove : ∠AOB = ∠COD.



Proof : In ΔAOB and ΔCOD,

OA = OC [Radii of a circle]

OB = OD [Radii of a circle]

AB = CD [Given]

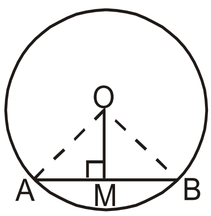
ΔAOB ΔCOD [By SSS congruency]

∠AOB = ∠COD. [By CPCT]

Converse :

If the angles subtended by the chords of a circle at the centre are equal, then the chords are equal.

Theorem : The perpendicular from the centre of a circle to a chord bisects the chord.



Given : A circle with centre O. AB is a chord of this circle.

To Prove : MA = MB.

Construction : Join OA and OB.

Proof : In right triangles OMA and OMB,

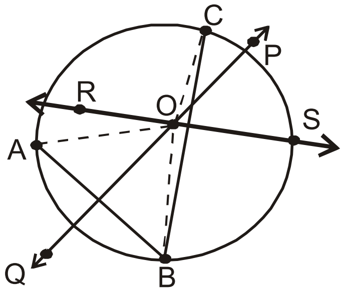
OA = OB	[Radii of a circle]
OM = OM	[Common]
$\angle$OMA = $\angle$OMB	[90º each]
OMA $\cong$ OMB	[By RHS]
MA = MB	[By cpct]

Converse :

The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.

Theorem : There is one and only one circle passing through three given non-collinear points.

Proof : Take three points A, B and C, which are not in the same line, or in other words, they are not collinear [as in figure]. Draw perpendicular bisectors of AB and BC say, PQ and RS respectively. Let these perpendicular bisectors intersect at one point O.(Note that PQ and RS will intersect because they are not parallel) [as in figure].



O lies on the perpendicular bisector PQ of AB. OA = OB

[Every point on the perpendicular bisector of a line segment is equidistant from its end points]

Similarly,

O lies on the perpendicular bisector RS of BC. OB = OC

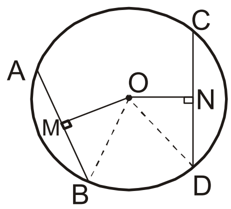
[Every point on the perpendicular bisector of a line segment is equidistant from its end points]

So, OA = OB = OC

i.e., the points A, B and C are at equal distances from the point O.

So, if we draw a circle with centre O and radius OA it will also pass through B and C. This shows that there is a circle passing through the three points A, B and C. We know that two lines (perpendicular bisectors) can intersect at only one point, so we can draw only one circle with radius OA. In other words, there is a unique circle passing through A, B and C.

Theorem : Equal chords of a circle (or of congruent circles) are equidistant from the centre (or centres).



Given : A circle have two equal chords AB & CD. i.e. AB = CD and OM $\perp$ AB, ON $\perp$ CD.

To Prove : OM = ON

Construction : Join OB & OD

Proof : AB = CD (Given)

[The perpendicular drawn from the centre of a circle to a chord bisects the chord]

AB = CD

BM = DN

In ΔOMB & ΔOND

$\angle OMB = \angle OND = 90^\circ$ [Given]

$OB = OD$ [Radii of same circle]

$BM = DN$ [Proved above]

$\Delta OMB \cong \Delta OND$ [By R.H.S. congruency]

$OM = ON$ [By CPCT]

REMARK :

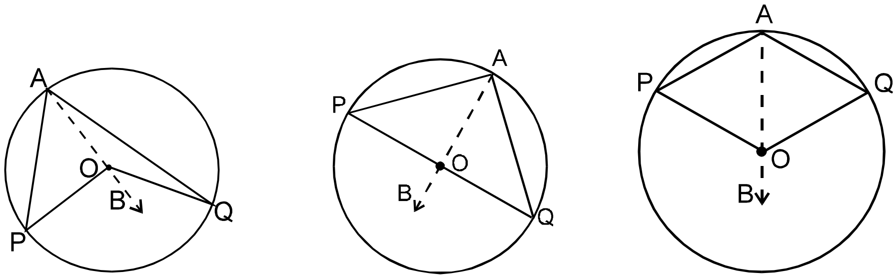
Chords equidistant from the centre of a circle are equal in length.

Theorem : The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.

Given : An arc PQ of a circle subtending angles POQ at the centre O and PAQ at a point A on the remaining part of the circle.

To Prove : $\angle POQ = 2\angle PAQ$.

Construction : Join AO and extend it to a point B.



(A) **(B)** **(C)**

Proof : There arises three cases:-

(A) arc PQ is minor

(B) arc PQ is a semi-circle

(C) arc PQ is major.

In all the cases,

$\angle BOQ = \angle OAQ + \angle AQO$ ...(i)

[An exterior angle of a triangle is equal to the sum of the two interior opposite angles]

In ΔOAQ,

$OA = OQ$ [Radii of a circle]

$\therefore \angle OQA = \angle OAQ$ ...(ii)

[Angles opposite equal sides of a triangle are equal]

From (i) and (ii)

$\angle BOQ = 2\angle OAQ$ ...(iii)

Similarly,

$\angle BOP = 2\angle OAP$ ...(iv)

Adding (iii) and (iv), we get

$\angle BOP + \angle BOQ = 2(\angle OAP + \angle OAQ)$

$\angle POQ = 2\angle PAQ.$ ...(v)

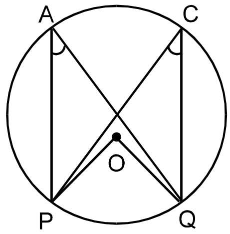
NOTE :

For the case (C), where PQ is the major arc, (v) is replaced by reflex angles.

Thus, reflex $\angle POQ = 2\angle PAQ$.

Theorem : Angles in the same segment of a circle are equal.

Proof : Let P and Q be any two points on a circle to form a chord PQ, A and C any other points on the remaining part of the circle and O be the centre of the circle. Then,



$\angle POQ = 2\angle PAQ$ ...(i)

And $\angle POQ = 2\angle PCQ$ ...(ii)

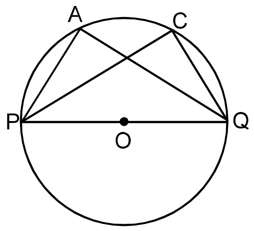
[Angle subtended at the centre is double than the angle subtended by it on the remaining part of the circle]

From equation (i) & (ii)

$2\angle PAQ = 2\angle PCQ$

$\angle PAQ = \angle PCQ$.

Theorem : Angle in the semicircle is a right angle.



Proof : $\angle PAQ$ is an angle in the segment, which is a semicircle.

$\therefore\ \angle PAQ = \frac{1}{2}\angle POQ = \frac{1}{2} \times 180^\circ = 90^\circ$ [POQ is straight line angle or $\angle POQ = 180^\circ$]

If we take any other point C on the semicircle, then again we get

$\angle PCQ = \frac{1}{2}\angle POQ = \frac{1}{2} \times 180^\circ = 90^\circ$.

Theorem : Let AB and DE be two chords of a circle with centre C. Suppose AB > DE. Then the distance from C to AB is less than the distance from C to DE.

As always, we draw the figure. Remember: by 'distance from C to AB' we mean the perpendicular distance. So, we drop perpendiculars CF and CG from C to AB and DE, respectively.

Given: $AB > DE$

To show that: $CF < CG$

Why is this true? In Fig. 5.16, both AC and CD are radii. So, $AC = CD$. By the Baudhāyana–Pythagoras Theorem,

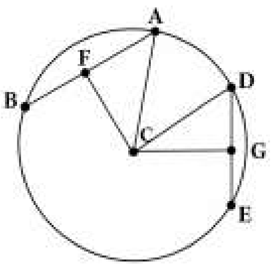
$CD^2 = CG^2 + GD^2$ and $AC^2 = CF^2 + AF^2$.

So, $CF^2 + AF^2 = CG^2 + GD^2$.

Now AB is greater than DE. So AF is greater than GD (because F and G are the midpoints of AB and DE).

Since $AF^2 > GD^2$, we have $CF^2 < CG^2$.

So, $CF < CG$.

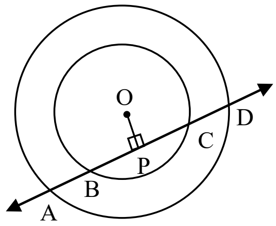


Comment: The chord nearest to the centre is the chord containing the centre. Its distance to C is zero! So the diameter is the greatest chord.

If one pushes the chord away from the centre, at some stage the chord becomes a single point, its length is zero, and the distance of C from this chord is the radius.

❖ EXAMPLES ❖

Ex.1 If a line intersects two concentric circles (circles with the same centre) with centre O at A, B, C and D. Prove that AB = CD (figure)



Sol. Draw $OP \perp AD$

For outer circle, AD is chord

$\therefore$ AP = PD .....(1) ($\because$ $\perp$ from centre bisect the chord)

& for inner circle, BC is chord

$\therefore$ BP = PC .....(2) ($\because$ $\perp$ from centre bisect the chord)

Subrtract equation (2) from eqation (1)

$\Rightarrow AP - BP = PD - PC \Rightarrow AB = CD$

Ex.2 If two circles intersect at two points, prove that their centres lie on the perpendicular bisector of the common chord.

Sol. **Given :** Two circles of radius r_1 & r_2 intersect at two different points A & B. and PQ is $\perp$ bisector of AB. $\therefore$ AO = OB & $\angle O = 90^\circ$

To prove : Line PQ is $\perp$ bisector of common chord AB.

Construction : Join A to C_1, C_2 and also B to C_1, C_2.

Proof : In ΔAC_1C_2 and ΔBC_1C_2

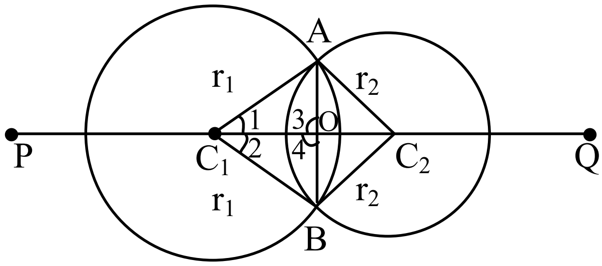
As, $C_1A = C_1B = r_1$

$C_1C_2 = C_1C_2$ (common)

$C_2A = C_2B = r_2$

$\Delta AC_1C_2 \cong \Delta BC_1C_2$ by SSS

$\therefore$ $\angle 1 = \angle 2$ (CPCT) ....(1)



Now in ΔAC_1O and ΔBC_1O

$C_1A = C_1B = r_1$

$\angle 1 = \angle 2$ (by 1)

$C_1O = C_1O$ (common)

$\Delta AC_1O \cong \Delta BC_1O$ by SAS

$\therefore$ AO = OC (CPCT) ....(2)

i.e PQ is bisector of line AB.

also $\angle 3 = \angle 4$ (CPCT).

But $\angle 3 + \angle 4 = 180^\circ$ (Linear pair of angles)

$\Rightarrow \angle 3 = \angle 4 = 90^\circ$ ...(3)

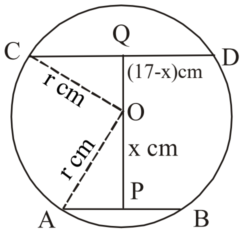
$\therefore$ $PQ \perp AB$

Hence PQ is $\perp$ bisector of common chord AB

$\therefore$ C_1 & C_2 lie on $\perp$ bisector of common chord.

Ex.3 AB and CD are two parallel chords of a circle such that AB = 10 cm and CD = 24 cm. If the chords are on the opposite sides of the centre and the distance between them is 17 cm, find the radius of the circle.

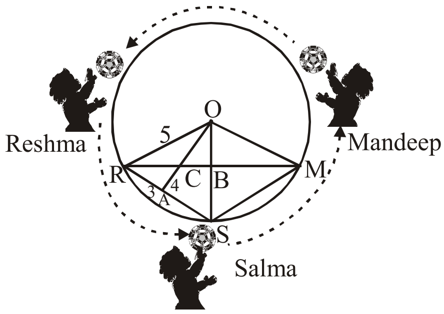
Sol. let O be the centre of the given circle and let its radius be r cm. Draw $OP \perp AB$ and $OQ \perp CD$. Since $OP \perp AB$, $OQ \perp CD$ and $AB \parallel CD$, therefore, points P, O and Q are collinear. So, PQ = 17 cm.
Let OP = x cm. Then, OQ = (17 – x) cm.
Join OA and OC. Then, OA = OC = r.



Since the perpendicular from the centre to a chord of the circle bisects the chord.
$\therefore$ AP = PB = 5 cm and CQ = QD = 12 cm.
In right triangles OAP and OCQ, we have
$OA^2 = OP^2 + AP^2$ and $OC^2 = OQ^2 + CQ^2$
$\Rightarrow \quad r^2 = x^2 + 5^2$ ....(i)
and, $\quad r^2 = (17 - x)^2 + 12^2$ ....(ii)
$\Rightarrow \; x^2 + 5^2 = (17 - x)^2 + 12^2$ [On equating the values of r^2]
$\Rightarrow \; x^2 + 25 = 289 - 34x + x^2 + 144 \; \Rightarrow \; 34x = 408 \Rightarrow x = 12$ cm.
Putting x = 12 cm in equation (i), we get $r^2 = 12^2 + 5^2 = 169 \Rightarrow r = 13$ cm.
Hence, the radius of the circle is 13 cm.

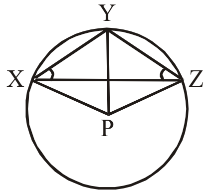
Ex.4 Three girls Reshma, Salma and Mandeep are playing a game by standing on a circle of radius 5 m drawn in a park. Reshma throws a ball to Salma, Salma to Mandeep. Mandeep to Reshma. If the distance between Reshma and Salma and between Salma and Mandeep is 6 m each. What is the distance between Reshma and Mandeep ?

Sol. Let the position of Reshma, Salma and Mandeep be at R, S and M on the circumference of the circular park.



RS = SM = 6 m
Radius = OR = OS = 5m.
In rt angled ΔRBO
$RB^2 = OR^2 - OB^2$ ....(1)
In rt angled ΔSBR
$RB^2 = RS^2 - SB^2$ ....(2)
From (1) and (2), we get
$OR^2 - OB^2 = RS^2 - SB^2 \quad \Rightarrow (5)^2 - x^2 = (6)^2 - (5 - x)^2$ (Let OB = x)
$25 - x^2 = 36 - x^2 - 25 + 10x \quad \Rightarrow \; 10x = -36 + 25 + 25$
$10x = 14 \Rightarrow x = \frac{14}{10}$
Distance between Reshma and Mandeep = RM
$= 2RB = 2\sqrt{OR^2 - OB^2} = 2\sqrt{25 - x^2} = 2\sqrt{25 - \left(\frac{14}{10}\right)^2}$
$= 2\sqrt{\frac{2500 - 196}{100}} = \frac{2\sqrt{2304}}{10} = 2 \times \frac{48}{10} = \frac{48}{5}$ m

Ex.5 P is the centre of the circle . Prove that $\angle XPZ = 2(\angle XZY + \angle YXZ)$.



Sol. **Given :** A circle with centre P, XY and YZ are two chords.

To prove : $\angle XPZ = 2(\angle XZY + \angle YXZ)$

Proof : In a circle with centre P, arc XY subtends $\angle XPY$ at the centre and $\angle XZY$ at remaining part of the circle.

$\Rightarrow \angle XPY = 2\angle XZY$ ....(1)

Similarly, arc YZ subtends $\angle YPZ$ at the centre and $\angle YXZ$ at remaining part

$\therefore \angle YPZ = 2\angle YXZ$ ....(2)

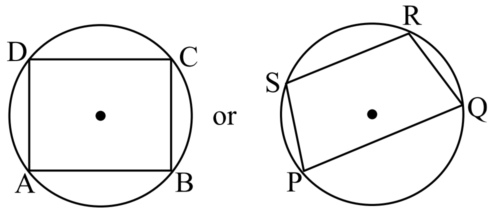
Adding (1), and (2), we get

$\angle XPY + \angle YPZ = 2\angle XZY + 2\angle YXZ$

$\Rightarrow \angle XPZ = 2(\angle XZY + \angle YXZ)$.

CONCYCLICITY AND CYCLIC QUADRILATERALS

If all four vertices of a quadrilateral are on the circumference of circle then it is called cyclic Quadrilateral.

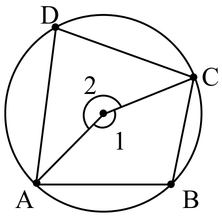


Properties:

1. **Sum of opposite angles** is 180º (or opposite angles of cyclic quadrilateral are supplementary)

Proof : For Arc $\overset{\frown}{ABC}$, $\angle D = \frac{1}{2}\angle 1$ .....(1)

& for Arc $\overset{\frown}{ADC}$, $\angle B = \frac{1}{2}\angle 2$ ......(2)



{angle at circumference is half of the angle at the centre}

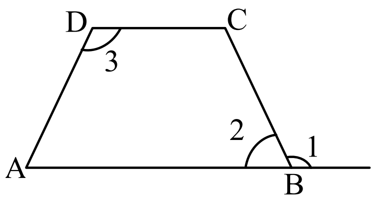
Now, adding equation (1) & (2)

$\angle B + \angle D = \frac{1}{2}(\angle 1 + \angle 2) = \frac{1}{2}(360) = 180^\circ$

2. **Exterior angle :** Exterior angle of cyclic quadrilateral is equal to opposite interior angle.

Proof : Let ABCD is cyclic quadrilateral.

Then $\angle 3 + \angle 2 = 180^\circ$ ....(1) ($\because$ opposite angle are supplementary)

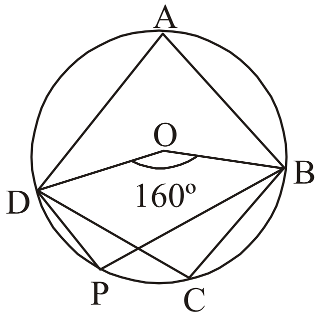


but $\angle 1 + \angle 2 = 180^\circ$ ....(2) (Linear pair of angles)

$\therefore$ from (1) & (2)

$\angle 3 + \angle 2 = \angle 1 + \angle 2 \Rightarrow \angle 3 = \angle 1$

Ex.6 In figure ABCD is a cyclic quadrilateral; O is the centre of the circle. If $\angle BOD = 160^\circ$, find the measure of $\angle BPD$, as shown in the given figure.



Sol. Consider the arc BCD of the circle. This arc makes angle $\angle BOD = 160^\circ$ at the centre of the circle and $\angle BAD$ at a point A on the circumference.

$\therefore \quad \angle BAD = \frac{1}{2}\angle BOD = 80^\circ$

Now, ABCD is a cyclic quadrilateral.

$\Rightarrow \quad \angle BAD + \angle BCD = 180^\circ$

$80^\circ + \angle BCD = 180^\circ$

$\Rightarrow \quad \angle BCD = 100^\circ$

$\Rightarrow \quad \angle BPD = 100^\circ$ [$\because$ $\angle BPD$ and $\angle BCD$ are angles in the same segment]

$\therefore \quad \angle BCD = \angle BPD$

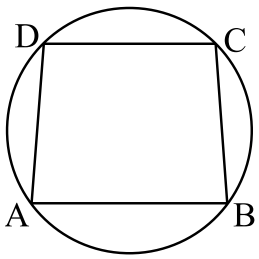
Ex.7 Prove that a cyclic parallelogram is a rectangle.

Sol. ABCD is cyclic

$\therefore \quad \angle A + \angle C = 180^\circ$ ...(1)

But ABCD is $||^{gm}$

$\therefore \quad \angle A = \angle C$ ....(2)



By (1) & (2)

$\angle A + \angle A = 180^\circ$

$2\angle A = 180^\circ$

$\angle A = 90^\circ$

$\therefore$ $||^{gm}$ ABCD is rectangle .