Measuring Space: Perimeter and Area

Chapter Outline

- Perimeter and Area of A Shape
- Heron's Formula
- Area of Quadrilateral using Heron's Formula
- Area of Polygons
- Area of Circles and Sectors

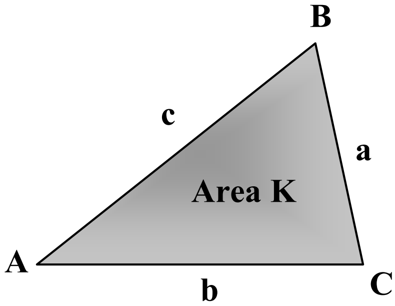
AREA OF TRIANGLE BY HERON'S FORMULA

Heron was born in about 10AD possibly in Alexandria in Egypt. His works on mathematical and physical subjects are so numerous and varied that he is considered to be an encyclopedic writer in these fields.

His geometrical works deal largely with problems on mensuration. He has derived the famous formula for the area of a triangle in terms of its three sides.

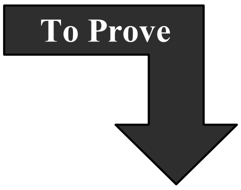
HERON (10AD - 75AD)

Heron's Formula



ΔABC : sides a, b, c

Semiperimeter. $s = \dfrac{a+b+c}{2}$



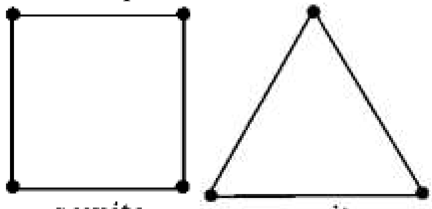
Area ΔABC: $K = \sqrt{s(s-a)(s-b)(s-c)}$

PERIMETER OF A SHAPE

Given any shape, its perimeter is the total length around its border. Imagine a tiny insect going for a walk around its border, never turning around, till it returns to its starting point. The perimeter of the shape is the total distance it travels.

A square with side a units has perimeter 4a units.

An equilateral triangle with side a units has perimeter 3a units.



The perimeter of a rectangle with length a units and width b units is 2(a + b) units

Perimeter of a Circle —The C/D Ratio

C/D's Adventurous Journey: From Ancient Approximations to the Exact Formula of Mādhava Mathematicians have been fascinated by circles, and the C/D ratio, since ancient times and across geographical regions. This constant, which we now call π (we say 'pi' as in 'pie' and not as in 'pizza'), represents a bridge between many different areas of mathematics, and connects the straight-edged world of polygons with the infinite curves of nature. While early civilisations relied on practical, 'good-enough' values for construction and trade, the pursuit of this ratio eventually became an important pastime of mathematicians that helped to benchmark the sophistication of mathematical theories.

Brahmagupta (628 CE) suggested the use of the value $\sqrt{10} \approx 3.1622$ for π ; while slightly less accurate, he chose it for its mathematical elegance and the ease with which it could be manipulated in equations, a preference that saw $\sqrt{10}$ become the dominant approximation in the Arab world and medieval Europe for centuries after.

Over the ensuing years, many great mathematicians studied π, including its approximations; but no approximation was as accurate as Zu's $\frac{355}{113}$ — or as algebraically simple as $\frac{22}{7}$, 3.1416, and $\sqrt{10}$ — until the work of Mādhava of Saṅgamagrāma, who changed the game forever.

Madhava realised that π was not just a number to be approximated by fractions and other finite algebraic expressions, but a limit to be reached. Mādhava thereby discovered the first exact formula for π:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

THE STORY OF π

π Is Irrational

The digits of π go on forever, with no visible pattern. You already know that fractions give rise to decimal expansions with a rhythmic pattern, e.g., $\frac{1}{3} = 0.33333\ldots, \frac{1}{11} = 0.09090909\ldots, \frac{1}{7} = 0.142857142857\ldots$

But for π , there is no such pattern! It turns out that π cannot be written as a ratio of two integers. Such numbers are called *irrational*.

Remember: A rational number is a number that can be written as a fraction $\frac{a}{b}$, where a, b are integers with $b \neq 0$, e.g., numbers such as $1\frac{2}{3}$, $\frac{-7}{11}$ and 1.4.

Historical Approximations:

Mesopotamia (c. 1900 BCE): Approximated π as 3.125

Archimedes (c. 250 BCE): Proved that π lies strictly between $3\frac{10}{71}$ and $3\frac{1}{7}$ using inscribed and circumscribed polygons.

Zu Chongzhi (480 CE): Discovered the "Close Ratio" $\frac{355}{113} \approx 3.1415929$, which remained the world's most accurate fractional approximation for over 800 years.

Aryabhata (499 CE): Provided the value 3.1416, notably describing it as *asanna* (approximate), showing profound insight into its nature.

Madhava of Sangamagrama: Birthed the early concepts of calculus by discovering the first exact infinite series formula for π: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Fun Fact

π is an Irrational Number. Its decimal digits go on forever without any repeating pattern.

π is **not** exactly equal to $\frac{22}{7}$; this fraction is merely a practical approximation ($\pi \approx \frac{22}{7}$).

Fun Fact: March 14 (3.14) is celebrated globally as Pi Day, and July 22 is celebrated as Pi Approximation Day.

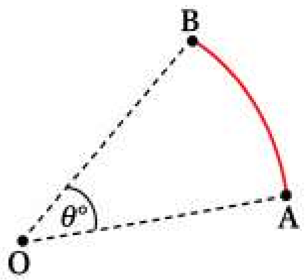
LENGTH OF AN ARC OF A CIRCLE

The circumference of a circle of diameter d is π**d**. Since the diameter is twice the radius, we can also write the circumference of a circle as **2πr,** where r is the radius.

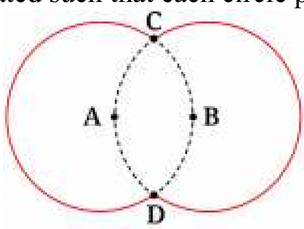
For semicircle = $2\pi r \div 2 = \pi r$

If the arc is **AB**, and it subtends an angle θ° at the centre **O** of the circle, the length of the arc is

$2\pi r \times \frac{\theta^\circ}{360^\circ}$.



Ex.1: Two circles of equal radius are located such that each circle passes through the centre of the other circle

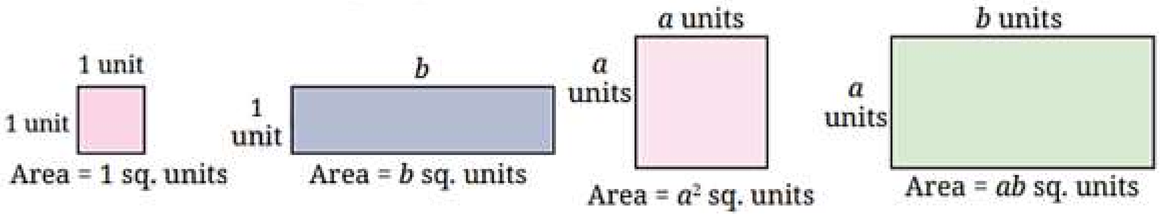


Sol. The triangle with vertices A, B, C. Since AB = r, AC = r, BC = r, the triangle is equilateral, hence $\angle CAB = 60° = \angle CBA$.

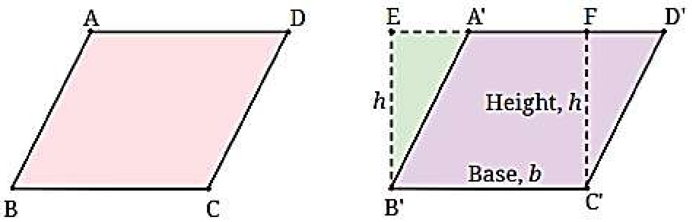
Similarly $\angle BAD = 60° = \angle ABD$. It follows that $\angle CAD = 120° = \angle CBD$.

Hence, each dotted arc is $\frac{1}{3}$ of the circumference of the circle on which it lies. Hence, the total length of the two red arcs is $2 \times \frac{2}{3} \times 2\pi r = \frac{8}{3}\pi r$.

AREA OF A RECTANGLE



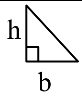
AREA OF A PARALLELOGRAM



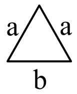
Area of the parallelogram is equal to base × height = bh.

AREA OF A TRIANGLE

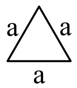
For right angled triangle = $\frac{1}{2} \times b \times h$



For isosceles triangle = $\frac{1}{4}\, b\sqrt{4a^2 - b^2}$



For equilateral triangle = $\frac{\sqrt{3}}{4}\, a^2$



A property of the median of a triangle

Theorem: A median of a triangle divides it into two triangles with equal area

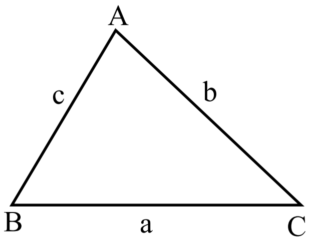
HERON'S FORMULA

If we have all sides of triangle and their is no way to find height then we use this formula for area of triangle.

$\boxed{\text{Area of } \Delta = \sqrt{s(s-a)(s-b)(s-c)}}$

Where s is semi perimeter of triangle.

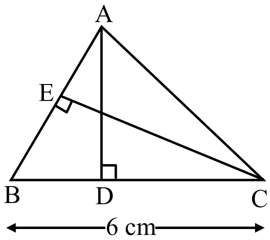
$s = \frac{1}{2}$ (sum of all sides) $= \frac{1}{2}(a + b + c)$



and a, b, c are sides of Δ.

Note : $\sqrt{2} = 1.41$, $\sqrt{3} = 1.73$, $\sqrt{5} = 2.23$,
$\sqrt{6} = 2.45$, $\sqrt{7} = 2.64$, $\sqrt{8} = 2.82$,
$\sqrt{11} = 3.31$, $\sqrt{15} = 3.87$

Ex.2 Find the length of AD in given figure, if EC = 4 cm and AB = 5 cm.



Sol. ∵ area of $\Delta ABC = \frac{1}{2}(AB \times EC) = \frac{1}{2}(5 \times 4) = 10$ square cm.

also area of $\Delta ABC = \frac{1}{2}(BC \times AD) = \frac{1}{2}(6 \times AD) = 3AD$ square cm

$\therefore \quad 3AD = 10 \quad \Rightarrow \quad AD = \frac{10}{3} = 3.33$ cm.

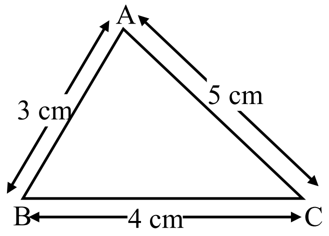
Ex.3 If perimeter of an equilateral triangle is 96 cm, then find its each side.

Sol. $\because$ Length of all sides are equal in equilateral Δ

Let length is x cm

$\therefore$ x + x + x = 96 $\Rightarrow$ 3x = 96 $\Rightarrow$ x = 32 cm.

Ex.4 For given figure find the value of s (s – a).



Sol. Perimeter = 2s = 3 + 4 + 5 = 12 cm

$\therefore$ semi perimeter = $\frac{12}{2}$ = 6 cm

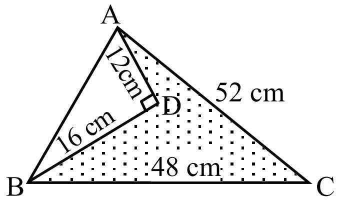
$\therefore$ s (s – a) = 6 (6 – 4) = 6 × 2 = 12 cm.

Ex.5 If one side from two equal sides of a Δ is 14 cm and semi-perimeter is 22.5 cm then find the third side.

Sol. Let the third side is x cm

$\therefore$ x + 14 + 14 = 2 × 22.5 $\Rightarrow$ x = 45 – 28 = 17 cm.

Ex.6 Find the area of the shaded region in figure :



Sol. By Pythagoras theorem, in ΔADB

$AB = \sqrt{AD^2 + BD^2} = \sqrt{12^2 + 16^2} = \sqrt{144 + 256} = \sqrt{400}$

AB = 20 cm.

$\therefore$ area of ΔABC $= \sqrt{s(s-a)(s-b)(s-c)}$ $\left\{s = \frac{20+48+52}{2} = 60\text{ cm}\right\}$

$= \sqrt{60(60-20)(60-48)(60-52)} = \sqrt{60 \times 40 \times 12 \times 8}$

$= \sqrt{(12 \times 5) \times (5 \times 8) \times 12 \times 8} = \sqrt{5^2 \times 8^2 \times 12^2} = 5 \times 8 \times 12 = 480\text{ cm}^2$.

also area of ΔADB = $\frac{1}{2}$ (AD) (BD) = $\frac{1}{2} \times 12 \times 16 = 96\text{ cm}^2$

$\therefore$ Shaded area = ar(ΔABC) – ar (ΔADB) = 480 – 96 = 384 cm^2.

Ex.7 A traffic signal board, indicating 'SCHOOL AHEAD', is an equilateral triangle with side 'a'. Find the area of the signal board, using Heron's formula. If its perimeter is 180 cm, what will be the area of the signal board ?

Sol. Let 2s be the perimeter of the signal board. Then, $2s = a + a + a \Rightarrow s = \frac{3a}{2}$

Let Δ be the area of the given equilateral triangle. Then,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \Rightarrow \Delta = \sqrt{\frac{3a}{2}\left(\frac{3a}{2}-a\right)\left(\frac{3a}{2}-a\right)\left(\frac{3a}{2}-a\right)} \qquad [\because a = b = c]$$

$$\Rightarrow \quad \Delta = \sqrt{\frac{3a}{2} \times \frac{a}{2} \times \frac{a}{2} \times \frac{a}{2}} = \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3}}{4}a^2$$

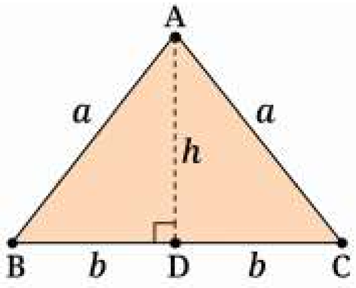
If, perimeter = 180 cm. Then, 2s = 180 $\Rightarrow$ 3a = 180 $\Rightarrow$ a = 60

$$\therefore \quad \Delta = \frac{\sqrt{3}}{4} \times (60)^2 = 900\sqrt{3}\text{ cm}^2.$$

Ex.8 An isosceles triangle with equal sides a units and base 2b units.

Sol. The semi-perimeter is $s=\frac{1}{2}(a+a+2b)=a+b$ units. Heron's formula now yields,

area of the triangle $=\sqrt{(a+b)(a+b-a)(a+b-a)(a+b-2b)}=b\sqrt{a^2-b^2}$ sq. units.



Let us check this against the 'half base times height' formula. Let h units be the height of the triangle.

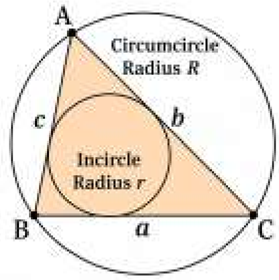
Then, by the Baudhāyana–Pythagoras theorem, $a^2-h^2=b^2$, $\therefore h^2=a^2-b^2$, $h=\sqrt{a^2-b^2}$.

So the area is $\frac{1}{2}\times\sqrt{a^2-b^2}\times 2b=b\sqrt{a^2-b^2}$ sq. units, the same as earlier.

Given any triangle ABC, there is exactly one circle that passes through its three vertices. This is called the **circumcircle** of ΔABC.

There is also exactly one circle that fits tightly in the triangle, touching its three sides. This is called the **incircle** of ΔABC.

Let the sides of ΔABC be a, b, and c. Let the radius of the circumcircle be R. Let the radius of the incircle be r. Then we have the following two formulas for the area:

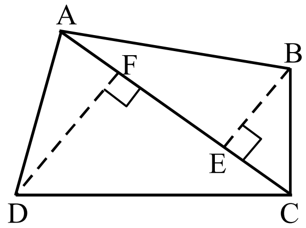


Area of $\Delta ABC=\frac{abc}{4R}$

Area of $\Delta ABC=\frac{r(a+b+c)}{2}$

AREA OF QUADRILATERAL

- Let ABCD be a quadrilateral, and AC be one of its diagonals. Draw perpendiculars BE and DF from B and D respectively to AC.



Area of quadrilateral ABCD = Area of ΔABC + Area of ΔADC

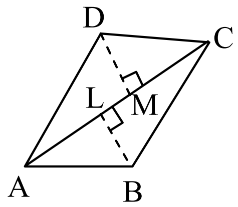
$$=\frac{1}{2}\ AC\times BE+\frac{1}{2}\ AC\times DF=\frac{1}{2}\ AC\ (BE+DF)$$

If AC = d, BE = h_1 and DF = h_2 then

Area of quadrilateral = $\frac{1}{2}d\ (h_1+h_2)$

Ex.9 In a four-sided field, the length of the longer diagonal is 28 m. The lengths of the perpendiculars from the opposite vertices upon this diagonal are 22.7 m and 17.3 m. Find the area of the field.

Sol. Let ABCD be the field, and let AC be its longer diagonal.



Let BL ⊥ AC and DM ⊥ AC.

Then, AC = 128 m, BL = 22.7 m and DM = 17.3 m

$\therefore$ area of the field = $\left\{\frac{1}{2} \times \text{AC} \times (\text{BL} + \text{DM})\right\}$ m^2 = $\left[\frac{1}{2} \times 128 \times (22.7 + 17.3)\right]$ m^2

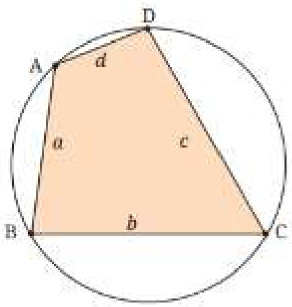
= (64 × 40) m^2 = 2560 m^2.

◈ If all four sides and a diagonal are given then by the diagonal, we get two triangles. By Heron's formula, we can find area of both triangles and by adding them, we get area of quadrilateral.

Brahmagupta's Formula for the Area of a Cyclic 4-gon

The formula states that if the sides of the cyclic 4-gon have lengths a, b, c, d, and the semi-perimeter is $s = \frac{1}{2}(a + b + c + d)$, then :

Area of 4-gon = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

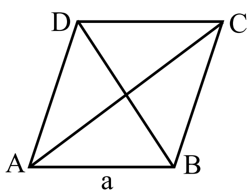


It is an amazing and beautiful formula!

Does it not remind us of Heron's formula for the area of a triangle?

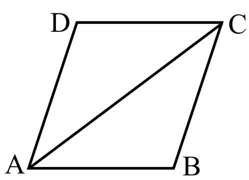
AREA OF RHOMBUS

◈ If both diagonals are given (or we can find their length) then area = $\frac{1}{2}$ × (Product of diagonals)



area = $\frac{1}{2}$(AC×DB) sq. units

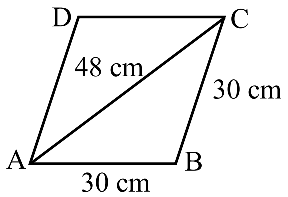
◈ If we use Heron's formula then we find area of one triangle made by two sides and a diagonal then twice of this area is area of rhombus.



area = 2× (ΔABC) sq. units

Ex.10 A rhombus shaped field has green grass for 18 cows to graze. If each side of the rhombus is 30 m and its longer diagonal is 48 m, how much area of grass field will each cow be getting ?

Sol.



s = (30 + 30 + 48)/2 = (108)/2 = 54 m

area = 2(ΔABC)

$= 2\sqrt{s(s-a)(s-b)(s-c)}$

$= 2\sqrt{54(54-30)(54-30)(54-48)}$

$= 2\sqrt{9\times6\times24\times24\times6}$

area = 2 × 3 × 6 × 24

∵ area required for 18 cows = (48 × 18) sq. units = 864 sq. units

∴ area required for 1 cow $= \dfrac{48\times18}{18}$ = 48 sq. units

OR

$$\text{area} = 2\left[\frac{b}{4}\sqrt{4a^2-b^2}\right] = 2\times\frac{48}{4}\sqrt{4(30)^2-(48)^2}$$

$$= 24\sqrt{3600-2304} = 24\sqrt{1296}$$

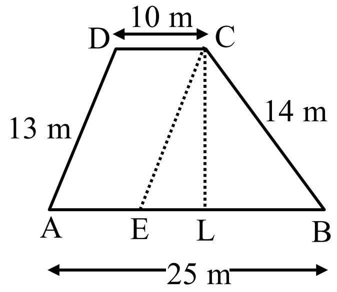
= 24 × 36 = 864 sq. units.

∴ area required for 1 cow $= \dfrac{48\times18}{18}$ = 48 sq. units

AREA OF TRAPEZIUM

Ex.11 A field is in the shape of a trapezium whose parallel sides are 25 m and 10 m. The nonparallel sides are 14 m and 13 m. Find the area of the field.

Sol. From C, draw CE || DA. Clearly, ADCE is a parallelogram having AD || CE and DC || AE such that AD = 13 m and DC = 10 m.



∴ AE = DC = 10 m and CE = AD = 13 m

⇒ BE = AB – AE = (25 – 10) m = 15 m

Thus in BCE, we have BC = 14 m, CE = 13 m and BE = 15 m

Let s be the semi-perimeter of ΔBCE. Then, 2s = BC + CE + BE = 14 + 13 + 15 = 42

⇒ s = 21

∴ Area of ΔBCE $=\sqrt{21\times(21-14)\times(21-13)\times(21-15)}$

⇒ Area of ΔBCE $=\sqrt{21\times7\times8\times6}$

⇒ Area of ΔBCE $=\sqrt{7^2\times3^2\times4^2} = 84\,m^2$

Also, Area of ΔBCE $=\dfrac{1}{2}$(BE × CL)

$\Rightarrow \quad 84 = \frac{1}{2} \times 15 \times CL$

$\Rightarrow \quad CL = \frac{168}{15} = \frac{56}{5}$

$\Rightarrow$ Height of parallelogram ADCE = CL = $\frac{56}{5}$ m

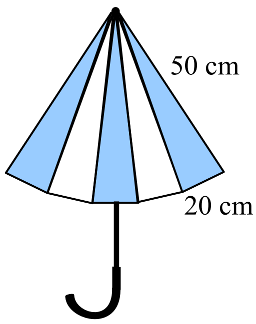
$\therefore$ Area of parallelogram ADCE

= Base × Height = AE × CL = $10 \times \frac{56}{5} = 112 \text{ m}^2$

Hence, Area of trapezium ABCD = Area of parallelogram ADCE + Area of ΔBCE

= $(112 + 84) \text{ m}^2 = 196 \text{ m}^2$.

Ex.12 An umbrella is made by stitching 10 triangular pieces of cloth of two different colours (see figure), each piece measuring 20 cm, 50 cm and 50 cm. How much cloth of each colour is required for the umbrella?



Sol. Sides of one triangular piece of cloth are of lengths a = 20 cm, b = 50 cm and c = 50 cm

Let s be the semi-perimeter of the triangular piece.

Then, $2s = a + b + c \Rightarrow 2s = 20 + 50 + 50 \Rightarrow s = 60$

$\therefore \Delta$ = Area of one triangular piece

$= \sqrt{s(s-a)(s-b)(s-c)}$

$\Rightarrow \Delta = \sqrt{60 \times (60-20) \times (60-50) \times (60-50)} \text{ cm}^2$

$\Rightarrow \Delta = \sqrt{60 \times 40 \times 10 \times 10} \text{ cm}^2 = \sqrt{6 \times 4 \times 10 \times 10 \times 10 \times 10} \text{ cm}^2 = 200\sqrt{6} \text{ cm}^2$

Area of cloth of each colour = $5 \times 200\sqrt{6} \text{ cm}^2 = 1000\sqrt{6} \text{ cm}^2$

AREA OF A CIRCLE

Area = πr^2 or $\frac{\pi d^2}{4}$

Minor sector : A sector of a circle is called a minor sector if the minor arc of the circle is a part of its boundary

In Fig. sector OAB is the minor sector.

Major sector : A sector of a circle is called a major sector if the major arc of the circle is a part of its boundary.

In Fig. sector OACB is the major sector.

❖ **Area of a Sector**

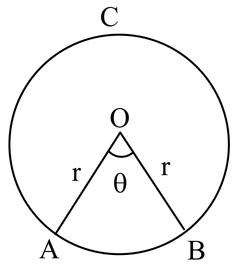
If the arc subtends an angle θ at the centre, then arc length is

$\ell = \frac{\theta}{360} \times 2\pi r = \frac{\theta}{180} \times \pi r$

Thus, the area of a sector of angle θ in a circle of radius r is given by

$A = \frac{\theta}{360} \times \pi r^2$

$= \frac{\theta}{360} \times$ (Area of the circle)



Ex.13 A sector is cut from a circle of radius 21 cm. The angle of the sector is 150º.Find the length of its arc and area.

Sol. The arc length l and area A of a sector of angle θ in a circle of radius r are given by

$l = \frac{\theta}{360} \times 2\pi r$ and $A = \frac{\theta}{360} \times \pi r^2$ respectively.

Here, r = 21 cm and θ = 150

$\therefore\ l = \left\{\frac{150}{360} \times 2 \times \frac{22}{7} \times 21\right\}$ cm = 55 cm

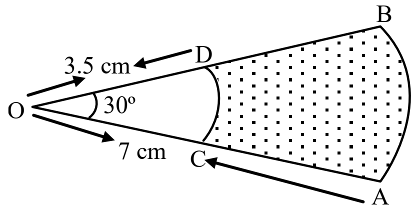
and $A = \left\{\frac{150}{360} \times \frac{22}{7} \times (21)^2\right\} \text{cm}^2 = \frac{1155}{2} \text{cm}^2 = 577.5 \text{ cm}^2$

Ex.14 In Fig. OCD & OAB are sectors of two concentric circles of radii 3.5 cm and 7 cm respectively. Find the area of the shaded region. (Use π = 22/7).

Sol. Let A_1 and A_2 be the areas of sectors OAB and OCD respectively. Then, A_1 = Area of a sector of angle 30º in a circle of radius 7 cm

$\Rightarrow A_1 = \left\{\frac{30}{360} \times \frac{22}{7} \times 7^2\right\} \text{cm}^2$ $\left[\text{Using: } A = \frac{\theta}{360} \times \pi r^2\right]$

$\Rightarrow A_1 = 77/6 \text{ cm}^2$



A_2 = Area of a sector of angle 30º in a circle of radius 3.5 cm.

$\Rightarrow A_2 = \left\{\frac{30}{360} \times \frac{22}{7} \times (3.5)^2\right\} \text{cm}^2$

$\Rightarrow A_2 = \left\{\frac{1}{12} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}\right\} \text{cm}^2 = \frac{77}{24} \text{cm}^2$

$\therefore$ Area of the shaded region $= A_1 - A_2 = \left(\frac{77}{6} - \frac{77}{24}\right) \text{cm}^2$

$= \frac{77}{24} \times (4 - 1) \text{ cm}^2 = 77/8 \text{ cm}^2 = 9.625 \text{ cm}^2$

Eg. Using Baudhayana's algebraic identity, show how a rectangle with sides of 8 cm and 2 cm can be mathematically translated into a square of equal area.

Sol. Baudhayana's method utilizes the algebraic identity: $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$

Let the sides be $a = 8$ and $b = 2$. The target area of the rectangle is $8 \times 2 = 16$ sq cm.

Applying the identity: $\left(\frac{8+2}{2}\right)^2 - \left(\frac{8-2}{2}\right)^2$

Simplify the terms: $5^2 - 3^2$.

Calculate the squares: $25 - 9 = 16$.