MIND MAP

Complex Number

1. Real & Imaginary Numbers

In general, Complex, $z = x + iy$

where, $i = \sqrt{-1}$

$x \longrightarrow$ the real part of z $\longrightarrow$ [Re(z)]

$y \longrightarrow$ the imaginary part of z $\longrightarrow$ [Im(z)]

2. Iota *(i)*

'i' is an imaginary number $i = \sqrt{-1}$

Infact $i^{4n} = 1$, $n \in I$,

$i^{4n+1} = i^{4n}.i = 1.i = i$

$i^{4n+2} = i^{4n}.i^2 = (1)(-1) = -1$

$i^{4n+3} = i^{4n}.i^3 = 1.(-i) = -i$

$i^{4n+4} = i^{4n}.i^4 = (1)(1) = 1$

3. Representation of Complex No.s

Every complex number 'z' can be written as : $z = x + iy$ where $x, y \in R$

& $i = \sqrt{-1}$

Every complex number can be represented as a set of ordered pair (x, y).

Classification

Every complex can be classified as :

$z = x + iy$

Purely real	Purely imaginary	Imaginary
if y = 0	if x = 0	if y ≠ 0

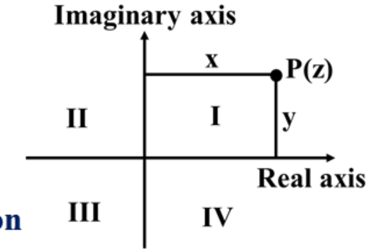
$0 + 0i$ is both a purely real as well as purely imaginary but not imaginary.

4. Argand Plane

Argand plane is divided in 4 quadrants by two axes. (i) Real axis (ii) Imaginary axis

All complex no. lying on the real axis were called as purely real.

And those lying on imaginary axis as purely imaginary.



$z = x + iy$

Re(z) = x, Im(z) = y

5. Algebra of Complex Numbers

Addition

$(a + ib) + (c + id) = a + c + ib + id = (a + c) + i(b + d)$

Subtraction

$(a + ib) - (c + id) = a - c + ib - id = (a - c) + i(b - d)$

Multiplication

$(a + ib)(c + id) = ac + i^2bd + iad + ibc = (ac - bd) + i(ad + bc)$

Division
Rationalization

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id}\cdot\frac{c-id}{c-id}$$

$$= \frac{ac - iad + ibc - i^2bd}{c^2 - \cancel{icd} + \cancel{idc} - i^2d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Equality

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ For Equality,

$z_1 = z_2$

$x_1 = x_2$ & $y_1 = y_2$

Re (z_1) = Re (z_2)

Im (z_1) = Im (z_2)

$a + ib > c + id$ has to be meaningful if,

$\Rightarrow b = d = 0 \therefore a > c$

We can compare real numbers,

But, cannot compare imaginary no.s

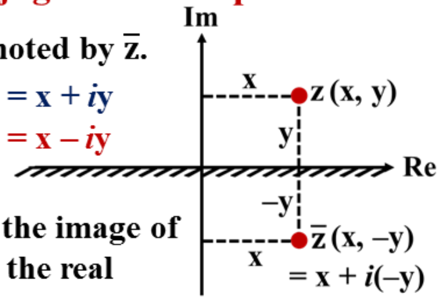
6. Conjugate of Complex No.

❖ Denoted by $\bar{z}$.

❖ If $z = x + iy$

$\rightarrow \bar{z} = x - iy$

❖ $\bar{z}$ is the image of z in the real axis.



$= x + i(-y)$

$= x - iy$

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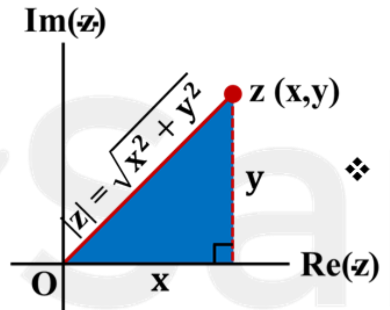
7. Properties of Conjugate If $z = x + iy$

1. $\overline{(\bar{z})} = z$
2. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ Similarly, $\overline{z_1 + z_2 + \ldots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n$
3. $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ Similarly, $\overline{z_1 - z_2 - \ldots - z_n} = \bar{z}_1 - \bar{z}_2 - \ldots - \bar{z}_n$
4. $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$ Similarly, $\overline{z_1 . z_2 . \ldots . z_n} = \bar{z}_1 . \bar{z}_2 . \ldots . \bar{z}_n$
5. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$; $z_2 \neq 0$
6. $z + \bar{z} = (x + \cancel{iy}) + (x - \cancel{iy})$
   $= 2x = 2\,\mathrm{Re}(z)$
   $\therefore \mathrm{Re}(z) = \dfrac{z + \bar{z}}{2}$
7. $z - \bar{z} = (x + \cancel{iy}) - (x - \cancel{iy})$
   $= 2iy = 2i\,\mathrm{Im}(z)$
   $\therefore \mathrm{Im}(z) = \dfrac{z - \bar{z}}{2i}$

Note:
If z is purely real $\therefore \mathrm{Im}(z) = 0$
$\Rightarrow z = \bar{z} \Rightarrow z - \bar{z} = 0$
If z is purely imaginary $\therefore \mathrm{Re}(z) = 0$
$\Rightarrow z = -\bar{z} \Rightarrow z + \bar{z} = 0$

Note: If $x + iy = f(a + ib)$ then,
$x - iy = f(a - ib)$
Further, $g(x + iy) = f(a + ib)$
$\Rightarrow g(x - iy) = f(a - ib)$

8. Modulus of Complex No.



$z = x + iy$

❖ Denoted by $|z|$.

❖ $|z| = \sqrt{x^2 + y^2} = \sqrt{\mathrm{Re}^2(z) + \mathrm{Im}^2(z)}$

❖ It denotes the shortest distance from the origin.

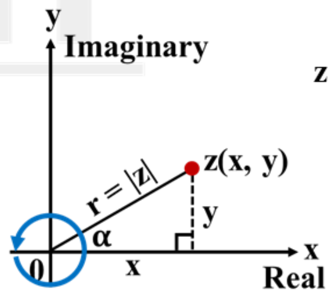
Properties of Modulus $z = x + iy$

1. $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
2. $|z| \geq 0$
   If $|z| = 0 \Rightarrow z$ is origin
3. $|z_1 z_2| = |z_1| . |z_2|$ Similarly,
   $|z_1 z_2 z_3 \ldots z_n| = |z_1| . |z_2| . |z_3| \ldots |z_n|$
   If $z_1 = z_2 = z_3 = \ldots = z_n = z$
   then, $|z^n| = |z|^n$
4. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$
5. $z.\bar{z} = |z|^2$

Note: If $|z| = 1$, then $z.\bar{z} = 1^2 = 1 \Rightarrow \bar{z} = \dfrac{1}{z}$

6. $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\,\mathrm{Re}(\bar{z}_1 z_2)$
7. $|z_1 + z_2|^2 + |z_1 - z_2|^2$
   $= 2\left[|z_1|^2 + |z_2|^2\right]$

9. Argument of Complex No.



$z = x + iy \equiv (x, y)$

Arg. of complex no. is inclination of directed line segment represented by a complex no. (z) with +ve real axis.

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9. Argument of Complex No.

(I) General Values of Argument

$= 2n\pi + \alpha, n \in N.$ $\alpha = \tan^{-1}\left(\frac{y}{x}\right)$

(II) Least Positive Argument

'α' is called least positive argument when it lies between $(0, 2\pi]$. $\boxed{\alpha \in (0, 2\pi]}$

(III) Amplitude (θ)

Also known as Principal value of argument of z. $\boxed{\theta \in (-\pi, \pi]}$

Important Properties of Amplitude

1. $\text{amp}(z_1 . z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi;\ k \in I$

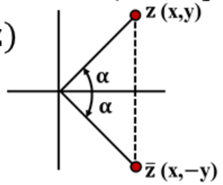
2. $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi;\ k \in I$

3. amp (z.z.z……z) n times = amp(z) + amp(z) + …… + amp(z) n times + $2k\pi;\ k \in I$

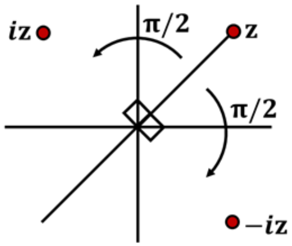
$\Rightarrow \text{amp}(z^n) = n\ \text{amp}(z)$

where proper value of 'k' must be chosen so that R.H.S. lies in $(-\pi, \pi]$.

4. $\text{amp}(\bar{z}) = -\text{amp}(z)$



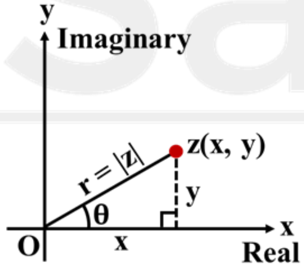
Note :



❖ $\text{amp}(iz) = \text{amp}(i) + \text{amp}(z) = \frac{\pi}{2} + \text{amp}(z)$

❖ $\text{amp}(-iz) = \text{amp}(-i) + \text{amp}(z) = -\frac{\pi}{2} + \text{amp}(z)$

10. Cartesian Form / Algebraic Form



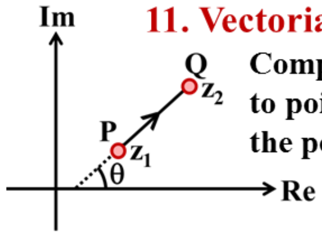
$z = x + iy$

$|z| = \sqrt{x^2 + y^2}$

$\bar{z} = x - iy$

$\text{amp}(z) = \theta$

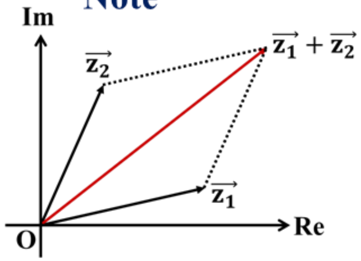
11. Vectorial Representation



Complex no. z corresponding to point A can be treated as the position vector of point A.

$\arg(z_2 - z_1) = \theta$

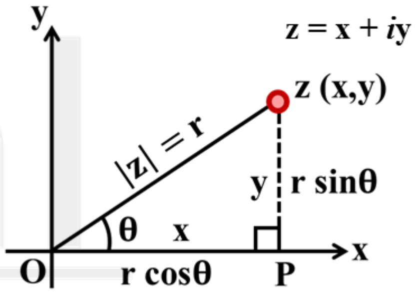
Note



By using parallelogram law of vector addition, Resultant = diagonal, i.e., $= \vec{z_1} + \vec{z_2}$

12. Trigonometric / Polar Form

$z = x + iy \equiv (x, y)$



$x = r\cos\theta$ $y = r\sin\theta$

$r = \sqrt{x^2 + y^2} = |z|$

$z = x + iy = r(\cos\theta + i\sin\theta)$

where $|z| = r$; $\text{amp}(z) = \theta$

13. Exponential / Euler Form

$e^{i\theta} = \cos\theta + i\sin\theta$ $z = r.e^{i\theta}$

$z = r(\cos\theta + i\sin\theta)$ $z = |z|.e^{i\theta}$

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14. De – Moivre's Theorem

If $z = (\cos\theta + i\sin\theta)$ then,

$z^n = (\cos\theta + i\sin\theta)^n$

- if $n \in I$ $\quad z^n = (\cos n\theta + i\sin n\theta)$
- if $n = \frac{p}{q}$ $\quad z^n = (\cos n\theta + i\sin n\theta)$
  (p & q $\in$ I & are coprime)
  It is one of the value of z^n

Basic Steps to determine the roots of a Complex Number

1. Write the complex no. whose roots are to be determined in polar form.
2. Add $2k\pi$ to the argument.
3. Apply De - Moivre's Theorem.
4. Put $k = 0, 1, 2, 3, \ldots, (n-1)$ to get all 'n' roots.

15. Cube Roots of Unity

$z = (1)^{1/3}$ Cubing both sides

$z^3 = 1$

$z^3 - 1^3 = 0$ *Applying Identity:*

$(z-1)(z^2+z+1) = 0$

$z - 1 = 0$ **OR** $z^2 + z + 1 = 0$

$z = 1$ $\qquad z = \dfrac{-1 \pm i\sqrt{3}}{2}$

Note :

a. The cube roots of unity are

$$1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2} = (1, \omega, \omega^2)$$

b. Product of Roots = 1

$z^3 = 1 \Rightarrow z^3 - 1^3 = 0$ **OR**

$$1.\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right).\left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right) = 1$$

$1.\omega.\omega^2 = 1 \Rightarrow \omega^3 = 1$

$\dfrac{1}{\omega} = \dfrac{\omega^2}{\omega^3} = \omega^2 \qquad \omega^{3n+1} = \omega$

$\qquad\qquad\qquad\qquad \omega^{3n+2} = \omega^2$

$\dfrac{1}{\omega^2} = \dfrac{\omega}{\omega^3} = \omega$

c. Sum of Roots = 0

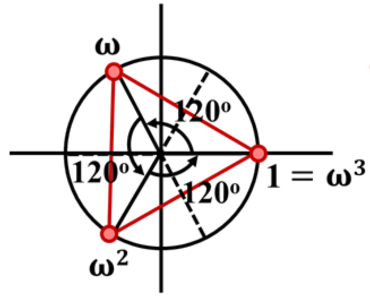
$z^3 = 1 \Rightarrow z^3 - 1^3 = 0$ **OR**

$1 + \omega + \omega^2 = 0$

$$1 + \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right) = 0$$

In general,

$$1 + \omega^r + \omega^{2r} = \begin{cases} 0 & \text{when } r \neq 3\lambda \\ 3 & \text{when } r = 3\lambda \end{cases}$$



d. $|1| = |\omega| = |\omega^2| = 1$

If any complex number 'z' is multiplied by ω, then 'z' rotates in ACW direction by an angle of 120°.

✓ When plotted on the argand plane, these roots constitute the vertices of an eq. Δ.

✓ This Δ lie on a circle of radius 1 units.

16. 'n' Roots of Unity

$$z = (1)^{1/n} = (\cos 0 + i\sin 0)^{1/n}$$

$$= [\cos(2m\pi) + i\sin(2m\pi)]^{1/n}$$

$$= \cos\left(\frac{2m\pi}{n}\right) + i\sin\left(\frac{2m\pi}{n}\right) = cis\left(\frac{2m\pi}{n}\right)$$

$$= e^{i\left(\frac{2m\pi}{n}\right)} = e^{im\theta} \quad m = 0, 1, 2, \ldots, (n-1)$$

$\& \ \theta = 2\pi/n$

z	$cis\left(\frac{2m\pi}{n}\right)$	$e^{i\left(\frac{2m\pi}{n}\right)}$	$e^{im\theta}$	Roots
z_1	$cis(0)$	e^0	e^0	1
z_2	$cis(2\pi/n)$	$e^{i\frac{2\pi}{n}}$	$e^{i\theta}$	α
z_3	$cis(4\pi/n)$	$e^{i\left(\frac{4\pi}{n}\right)}$	$e^{i2\theta}$	α^2
z_4	$cis(6\pi/n)$	$e^{i\left(\frac{6\pi}{n}\right)}$	$e^{i3\theta}$	α^3
$\vdots$				
z_n	$cis\frac{2(n-1)\pi}{n}$	$e^{i\left(2(n-1)\frac{\pi}{n}\right)}$	$e^{i(n-1)\theta}$	$\alpha^{(n-1)}$

G.P.

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16. ‘n’ Roots of Unity

$$\underbrace{1}_{1},\ \underbrace{\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}}_{\alpha},\ \underbrace{\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}}_{\alpha^2}$$

$$,\ldots,\ \underbrace{\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}}_{\alpha^{n-1}}$$

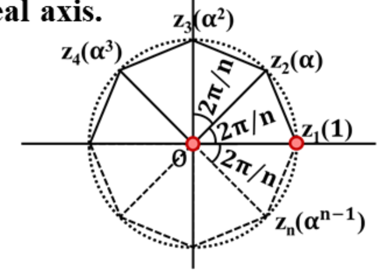
$i.e.,\ e^0, e^{i\theta}, e^{i2\theta}, e^{i4\theta}, e^{i6\theta} \ldots\ldots\ldots e^{i(n-1)\theta}$

where $\theta = \frac{2\pi}{n}$

$\Rightarrow$ n roots $\rightarrow 1, \alpha, \alpha^2, \ldots\ldots, \alpha^{n-1}$ are in G.P.

Note :

- The points represented by ‘n’ roots of unity are located at the vertices of a regular polygon of ‘n’ sides.
- This polygon inscribed in a unit circle (radius = 1) having centre at the origin.
- One vertex of this polygon being on the positive real axis.



(a) Sum of n – roots of $(1)^{1/n}$:

$z_1 + z_2 + \ldots + z_n = 1 + \alpha + \alpha^2 + \ldots \alpha^{n-1}$

➢ The sum of n roots of unity is zero.

(b) Product of Roots:

$z_1 . z_2 . z_3 \ldots z_n = 1.\alpha.\alpha^2.\alpha^3 \ldots \alpha^{n-1}$

$z_1 . z_2 . z_3 \ldots z_n =$

$1,\ n$ is odd $\longrightarrow (n-1)$ is even

$-1,\ n$ is even $\longrightarrow (n-1)$ is odd

(c) $1^k + \alpha^k + \alpha^{2k} + \alpha^{3k} + \ldots + \alpha^{(n-1)k} =$

$$= 1\left(\frac{1-\alpha^{kn}}{1-\alpha^k}\right) = \frac{1-\left(e^{i\frac{2\pi}{n}}\right)^{kn}}{1-\alpha^k} = \frac{1-(e)^{i2k\pi}}{1-(e)^{i2\frac{k\pi}{n}}}$$

$$= \frac{1-\{\cos(2k\pi) + i\sin(2k\pi)\}}{1-\left\{\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right\}}$$

$$= \begin{cases} \frac{0}{(\neq 0)} = 0 & \text{if } k \neq \lambda n \\ & k, \lambda \in I \\ \frac{0}{0} = \text{indeterminant} & \text{if } k = \lambda n \\ & k, \lambda \in I \end{cases}$$

17. Geometry Of Complex Number

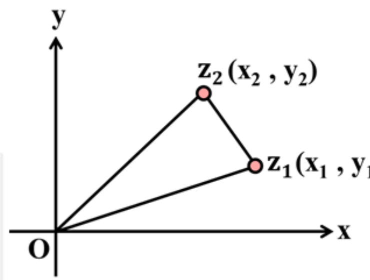
(i) Distance Formula:

$z_1 = x_1 + iy_1\ ;\ z_2 = x_2 + iy_2$

Distance between the two complex numbers z_1 & z_2 is given by $|z_1 - z_2|$

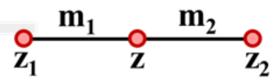
$\Rightarrow\ z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

$\Rightarrow |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$



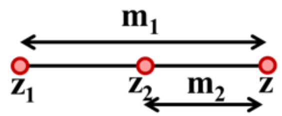
(ii) Section Formula:

(a)



$$z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$$ (Internal Division)

(b)



$$z = \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$$ (External Division)

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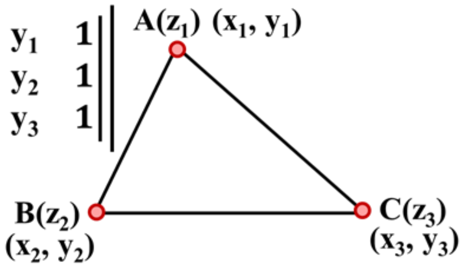
17. Geometry Of Complex Number

(iii) Area of Triangle:

$$\Delta = \left| \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$



For collinearity of 3 points: $\Delta = 0$

(iv) Centroid, Incentre, Orthocenter & Circumcenter of a triangle on a Complex Plane

(i) Centroid $G = \dfrac{z_1 + z_2 + z_3}{3}$

(ii) Incentre $I = \dfrac{az_1 + bz_2 + cz_3}{a + b + c}$

(iii) Orthocenter O

$$z_0 = \frac{z_1 \cdot \tan A + z_2 \cdot \tan B + z_3 \cdot \tan C}{\Sigma \tan A}$$

OR

$$= \frac{z_1 \cdot \tan A + z_2 \cdot \tan B + z_3 \cdot \tan C}{\Pi \tan A}$$

(iv) Circumcenter C'

$$z_{C'} = \frac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\Sigma \sin 2A}$$

Note :

In a triangle, *orthocenter* (O), *centroid* (G) & *circumcenter* (C') are collinear & G divides OC' in the ratio 2 : 1.

18. Rotation

Let the point P represents the complex number z_1 then,

$\overrightarrow{OP} = z_1 = r.e^{i\theta}$

α = Angle of rotation

$\overrightarrow{OQ} = z_2 = r.e^{i(\theta + \alpha)}$

$= r.e^{i\theta}.e^{i\alpha} = z_1.e^{i\alpha}$

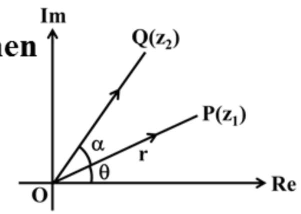
$z_2 = z_1.e^{i\alpha}$

Note : $z_2 = z_1.e^{i\alpha}$

(i) If $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are of unequal magnitude, then

$\widehat{OQ} = \widehat{OP}\, e^{i\alpha}$

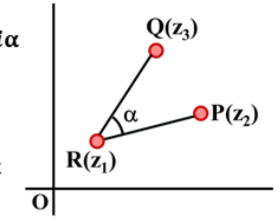
$$\frac{z_2}{|z_2|} = \frac{z_1}{|z_1|} e^{i\alpha}$$



(ii) $\widehat{RQ} = \widehat{RP}\, e^{i\alpha}$

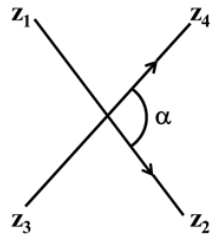
$$\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_2 - z_1}{|z_2 - z_1|} e^{i\alpha}$$

$$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| e^{i\alpha}$$



where $\alpha = \arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)$

(iii) $\dfrac{z_4 - z_3}{|z_4 - z_3|} = \dfrac{z_2 - z_1}{|z_2 - z_1|} e^{i\alpha}$



19. Equation Of Straight Line

Equation of a straight line through z_1 and z_2

If a variable complex number z(x, y) moves in such a way that z, z_1 & z_2 are collinear, then locus of 'z' describes a line through z_1 & z_2 .

$\overrightarrow{PA}$ & $\overrightarrow{PB}$ are always collinear, i.e.

$\arg(z - z_1) - \arg(z - z_2) = 0$ or π

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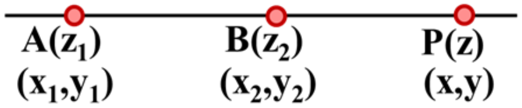
19. Equation Of Straight Line

or $\arg\left(\frac{z-z_1}{z-z_2}\right) = 0$ or π

i.e. $\frac{z-z_1}{z-z_2}$ is purely real

$$\boxed{\frac{z-z_1}{z-z_2} = \frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2}}$$



General Equation of Straight line

$Ax + By + C = 0 \quad m = \frac{-A}{B}$

$A.\frac{(z+\bar{z})}{2} + B.\frac{-i(z-\bar{z})}{2} + C = 0$

$z(A - iB) + \bar{z}(A + iB) + 2C = 0$

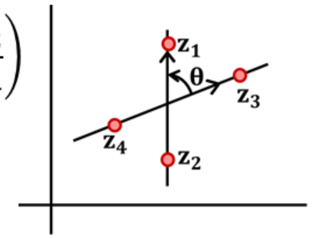
$$\boxed{\bar{\alpha}z + \alpha\bar{z} + r = 0}$$

where, $\alpha = A + iB$

Angle between lines:

$\theta = \arg(z_1 - z_2) - \arg(z_3 - z_4)$

$\theta = \arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right)$



Condition for lines to be parallel:

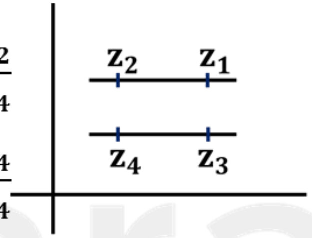
$\Rightarrow \quad m_1 = m_2 \qquad \arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = k\pi$

$\Rightarrow \quad \frac{z_1 - z_2}{z_3 - z_4}$ is purely real

$\Rightarrow \quad \frac{z_1 - z_2}{z_3 - z_4} = \frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_4}$

$\Rightarrow \quad \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} = \frac{z_3 - z_4}{\bar{z}_3 - \bar{z}_4}$

$\Rightarrow w_1 = w_2$



Condition for lines to be perpendicular:

$\Rightarrow \quad m_1 . m_2 = -1$

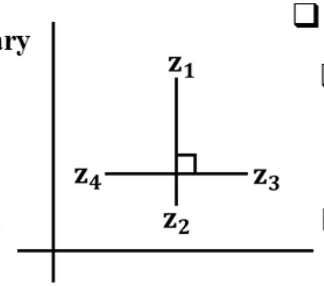
$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = (2n+1)\frac{\pi}{2}$

$\Rightarrow \quad \frac{z_1 - z_2}{z_3 - z_4}$ is purely imaginary

$\Rightarrow \quad \frac{z_1 - z_2}{z_3 - z_4} = -\frac{\bar{z}_1 - \bar{z}_2}{\bar{z}_3 - \bar{z}_4}$

$\Rightarrow \quad \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2} + \frac{z_3 - z_4}{\bar{z}_3 - \bar{z}_4} = 0$

$\Rightarrow \quad w_1 + w_2 = 0$



Parametric equation of a Line:

$\therefore \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} \quad z = z_1 + \lambda(z_2 - z_1)$

where $\lambda \in R$

20. Circle

Centre - Radius Form

Centre: $z_0 = (a + ib)$ Point: $z = (x + iy)$

Radius: r $\quad \boxed{|z - z_0| = r}$

Note :

A point P(z) moves in argand plane in such a way that its distances from two fixed points $A(z_1)$ & $B(z_2)$ are such that

$\frac{PA}{PB} = K \qquad \left|\frac{z - z_1}{z - z_2}\right| = k, \quad (k > 0)$

K = 1
Locus : *Straight Line*
(perpendicular bisector of line joining z_1 & z_2)

K ≠ 1
Locus : *Circle*

21. Triangle Inequality

❑ $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

❑ $||z_1| - |z_2|| \le |z_1 - z_2| \le |z_1| + |z_2|$

❑ $||z_1| - |z_2|| \le |z_1 \pm z_2| \le |z_1| + |z_2|$