MIND MAP

Binomial Theorem

1. Binomial Theorem

General Expansion

$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_n x^0 y^n$

General Term :

$T_{r+1} = {}^nC_r x^{n-r} y^r$ where, $0 \le r \le n$

1) The number of terms in the expansion of $(x+y)^n$ is $(n+1)$ i.e. one more than the index.

2) The sum of the indices of x & y in each term is n.

3) Power of first variable (x) decreases while of second variable (y) increases.

4) Binomial coefficients of the terms equidistant from the beginning and from the end are equal.

5) Binomial coefficients of the middle term is greatest.

6) m^{th} - Term from the END

T_m [from the end] $\longleftrightarrow$ T_{n-m+2} [from the beginning]

$\Rightarrow$ ${}^nC_r$ will be max

- where $r = \frac{n}{2}$, if n is even
- where $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$, if n is odd

2. Middle Term

Middle term in the expansion of $(I + II)^n$ is

$$\begin{cases} T_{\frac{n}{2}+1} & \text{when n is even} \\ T_{\frac{n+1}{2}} \ \& \ T_{\frac{n+3}{2}} & \text{when n is odd} \end{cases}$$

In binomial expansion, middle term has greatest binomial coefficient and if there are 2 middle terms, their coefficients will be equal.

3. Number of Terms in Expansion

(a) If n is ODD, then number of terms in

$(x+a)^n \pm (x-a)^n$ is $\frac{n+1}{2}$

(b) If n is EVEN, then number of terms in

(i) $(x+a)^n + (x-a)^n$ is $\frac{n}{2}+1$

(ii) $(x+a)^n - (x-a)^n$ is $\frac{n}{2}$

4. Numerically Greatest Term in the expansion of $(a+bx)^n$

$$\boxed{\left(\frac{n+1}{1+\left|\frac{I}{II}\right|}\right) - 1 \le r \le \left(\frac{n+1}{1+\left|\frac{I}{II}\right|}\right)}$$

5. Standard Binomial Expansion

$$\boxed{(1+x)^n = C_0x^0 + C_1x^1 + C_2x^2 + \dots\dots + C_nx^n}$$

Note : Binomial coefficient & Coefficient of x^r are equal

6. Properties of Binomial Coefficients & Summation of Series

$$\boxed{\sum_{r=0}^{n} {}^nC_r = 2^n} \qquad \boxed{\sum_{r=0}^{n} (-1)^r \, {}^nC_r = 0}$$

$$\boxed{C_0+C_2+C_4+\dots = C_1+C_3+C_5+\dots = 2^{n-1}}$$

- ${}^nC_1 + 2.{}^nC_2 + 3.{}^nC_3 + \dots\dots\dots + (n-1).{}^nC_{n-1} + n.{}^nC_n = n.\,2^{n-1}$
- $(1)^2.C_1 + (2)^2.C_2 + (3)^2.C_3 + \dots\dots\dots + (n)^2.C_n = n(1+n)\,2^{n-2}$
- $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \dots\dots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$
- $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \dots\dots + (-1)^n \frac{C_n}{n+1} = \frac{1}{n+1}$
- $C_0^2 + C_1^2 + C_2^2 + C_3^2 + \dots\dots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!\,n!}$
- $C_0C_r + C_1C_{r+1} + \dots\dots + C_{n-r}C_n = {}^{2n}C_{n-r} = \frac{(2n)!}{(n+r)!\,(n-r)!}$

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6. Properties of Binomial Coefficients & Summation of Series

- $C_0C_1 + C_1C_2 + C_2C_3 + ... + C_{n-1}C_n = {}^{2n}C_{n-1} = \dfrac{(2n)!}{(n+1)!\,(n-1)!}$
- $C_0C_2 + C_1C_3 + C_2C_4 + ... + C_{n-2}C_n = {}^{2n}C_{n-2} = \dfrac{(2n)!}{(n+2)!\,(n-2)!}$
- $C_0^2 - C_1^2 + C_2^2 + ... + (-1)^n C_n^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2}\,{}^nC_{n/2} & \text{if } n \text{ is even} \end{cases}$

7. Sum of Coefficients in an Expansion

- Sum of all the coefficients in an expansion can be obtained by putting variables = 1.
- Sum of coefficients of EVEN power of x can be obtained by

$$\frac{(\text{Put variables} = 1) + (\text{Put variables} = -1)}{2}$$

- Sum of coefficients of ODD power of x can be obtained by

$$\frac{(\text{Put variables} = 1) - (\text{Put variables} = -1)}{2}$$

8. Application of Binomial Theorem

Divisibility Problems

$x^n - y^n$ is always divisible by $(x - y)$

9. Multinomial Theorem

$$(x+y)^n = \sum_{r+s=n}^{n} \frac{n!}{s!\,r!} x^s y^r \quad \text{Where, } s + r = n$$

$$(x+y+z)^n = \sum_{r+s+t=n}^{n} \frac{n!}{s!\,r!\,t!} x^s y^r z^t \quad \text{Where, } s + r + t = n$$

This result can be generalized in the following form

$$(x_1 + x_2 + ... + x_k)^n = \sum_{r_1+r_2+.....+r_k=n}^{n} \frac{n!}{r_1!\,r_2!.....r_k!} x_1^{r_1} x_2^{r_2} .... x_k^{r_k}$$

where,

(i) $r_1 + r_2 + r_3 + ... + r_k = n$

(ii) $r_1, r_2, r_3, ...\ r_k \in w$

(iii) $n \in N$

(iv) The general term in the above expansion

$$= \frac{n!}{r_1!\,r_2!\,r_3!.....r_k!} \cdot x_1^{r_1} x_2^{r_2} x_3^{r_3} ......x_k^{r_k}$$

(v) Number of terms = $\boxed{{}^{n+k-1}C_{k-1}}$

10. Binomial Theorem for Negative & Fractional Index

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ... + \infty$$

where,

(i) n is a negative integer or a fraction.

(ii) $|x| < 1$

General term in the expansion of $(1 + x)^n$ is :

$$T_{r+1} = \frac{n(n-1)(n-2).............[n-(r-1)]}{r!} x^r$$

11. Important expansions for $n \in Q$ & $|x| < 1$

(a) $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + ... + \infty$

(b) $(1+x)^{-n} = 1 + (-n)x + \dfrac{(-n)(-n-1)}{2!}x^2 + \dfrac{(-n)(-n-1)(-n-2)}{3!}x^3 + ... + \infty$

(c) $(1-x)^n = 1 + n(-x) + \dfrac{n(n-1)}{2!}(-x)^2 + \dfrac{n(n-1)(n-2)}{3!}(-x)^3 + ... + \infty$

(d) $(1-x)^{-n} = 1 + (-n)(-x) + \dfrac{(-n)(-n-1)}{2!}(-x)^2 + \dfrac{(-n)(-n-1)(-n-2)}{3!}(-x)^3 + ...\infty$