MIND MAP

Determinant

1. Introduction

Value of the Determinant

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad D = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$$

$D = a_1b_2 - a_2b_1 \quad D = 2 \times 4 - 3 \times 1$

$= 8 - 3 = 5$

Note A determinant of order 1 is the number itself. $\Rightarrow |x| = x$

Expanding the Determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

This is know as row-wise expansion of the determinant.

2. General Representation of Determinant

A general third order determinant is represented by :

Note

$$D = \begin{vmatrix} a_{11} & \bar{a}_{12} & a_{13} \\ \bar{a}_{21} & a_{22} & \bar{a}_{23} \\ a_{31} & \bar{a}_{32} & a_{33} \end{vmatrix}$$

In the similar fashion we can expand a determinant in 6 different ways using elements of R_1, R_2, R_3, C_1, C_2 & C_3.

3. Minors and Cofactors

Minors

General third order determinant :

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minor of elements :

$$a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Minor of element $a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

General third order determinant :

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Hence, Minor of an element is defined as the minor determinant obtained by deleting a particular row and column in which that element lies.

Cofactors

General third order determinant :

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Cofactor of any element a_{ij} of determinant D is given by :

$$C_{ij} = (-1)^{i+j}.M_{ij}$$

$\therefore$ Cofactor is nothing but minor with a proper sign.

$C_{12} = (-1)^{1+2}.M_{12} = (-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$C_{ij} = (-1)^{i+j}.M_{ij}$

$C_{22} = (-1)^{2+2}.M_{22} = (1) \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

We know that,

$D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

$D = a_{11}M_{11} + a_{12}(-M_{12}) + a_{13}M_{13}$

$\Rightarrow D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Key Point

1. Sum of product of elements of any row (column) with their corresponding cofactors is equal to the value of DETERMINANT.

$$\Rightarrow \sum_{j=1}^{n} a_{ij}C_{ij} = D$$

Key Point

2. Sum of product of elements of any row (column) with cofactors of corresponding elements of any other row (column) is ZERO.

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Rightarrow a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$$

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \\ 5 & 7 & 8 \end{vmatrix} = 38$$

$C_{11} = -42$ $\quad C_{12} = -2$ $\quad C_{13} = 28$

$C_{21} = 5$ $\quad C_{22} = -7$ $\quad C_{23} = 3$

$C_{31} = 12$ $\quad C_{32} = 6$ $\quad C_{33} = -8$

$$D_c = \begin{vmatrix} -42 & -2 & 28 \\ 5 & -7 & 3 \\ 12 & 6 & -8 \end{vmatrix} = (38)^2$$

3. $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $D_C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$

Then, $D_C = D^2$

$\therefore$ In general, $D_C = D^{n-1}$

Where 'n' is the order of determinant.

MIND MAP

Determinant

4. Properties of Determinants

Property - 1

1. The value of a determinant remains unchanged, if the rows & columns are interchanged.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D^T$$

$C_1 \to R_1 \quad C_2 \to R_2 \quad C_3 \to R_3$

Property - 2

2. If any two rows (or columns) of a determinant are interchanged, the value of determinant is changed in sign only.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = -D$$

Property - 3

3. If a determinant has any two rows (or columns) identical, then its value is ZERO.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix} \quad \text{Here, } R_2 = R_3$$

$$= a_1 \times (\cancel{b_2c_2} - \cancel{b_2c_2}) - b1 \times (\cancel{a_2c_2} - \cancel{a_2c_2})$$
$$+ c1 \times (\cancel{a_2b_2} - \cancel{a2b2}) = 0$$

Property - 4

4. If all the elements of any row (or column) are multiplied by the same number, then value of the determinant is multiplied by that number.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \ \& \ D' = \begin{vmatrix} Ma_1 & Mb_1 & Mc_1 \\ a_2 & b_2 & c_2 \\ Na_3 & Nb_3 & Nc_3 \end{vmatrix}$$

Then, $D' = M.N.D$

Property - 5

5. If each element of any row (or column) can be expressed as a sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

$$D = \begin{vmatrix} a+x & b+y \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} x & y \\ c & d \end{vmatrix}$$

$$= (a+x)d - c(b+y) \quad (ad - bc) + (xd - yc)$$
$$= (ad - bc) + (xd - yc)$$

MIND MAP

Determinant

Property - 6

6. The value of a determinant is not changed by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).

5. Special Determinants

1. $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

2. $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

3. $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

4. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= -\frac{1}{2}(a+b+c).\{(a-b)^2+(b-c)^2+(c-a)^2\}$

6. Factor Theorem

If by putting x = a the value of a determinant vanishes then, (x – a) will be a factor the determinant. This is known as FACTOR THEOREM.

7. Multiplication of Determinants

Introduction

$D_1 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $D_2 = \begin{vmatrix} p & q \\ r & s \end{vmatrix}$

$D = D_1 \times D_2$

$D = \begin{vmatrix} \boxed{ap+br} & \boxed{aq+bs} \\ \boxed{cp+dr} & \boxed{cq+ds} \end{vmatrix}$

(Row by column multiplication)

Note

Multiplication can also be done

1) Row by Row
2) Column by Row
3) Column by Column

8. Cramer's Rule System of Linear Equations

- Types of solutions
  - Solution exist (At least one solution) (Consistent)
    - Unique/ Exactly one sol.
      - All variables zero is the only solution $x = 0, y = 0, z = 0$
        - Trivial solution
      - At least one non – zero variable satisfy the system
        - Non - Trivial solution
        - Non - Zero solution
    - Infinite solutions
  - No solution (Inconsistent)

9. Non – Homogeneous System

$a_1x + b_1y + c_1z = d_1$ .... (i)

$a_2x + b_2y + c_2z = d_2$ .... (ii)

$a_3x + b_3y + c_3z = d_3$ .... (iii)

$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Then, $x = \frac{D_x}{D}$ $y = \frac{D_y}{D}$ $z = \frac{D_z}{D}$

This is known as the CRAMER'S RULE

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9. Non – Homogeneous System

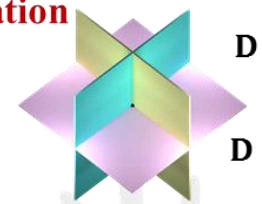
D	D_x, D_y, D_z	Type of Solution	Type of System
$\neq 0$	Atleast any one of $D_x, D_y, D_z \neq 0$	Unique Non – Trivial	Consistent
$\neq 0$	$D_x = D_y = D_z = 0$	Unique / Trivial	Consistent
$= 0$	$D_x = D_y = D_z = 0$	Infinite Sols.	Consistent
$= 0$	$D_x \neq 0$, or $D_y \neq 0$, or $D_z \neq 0$	No Solution	Inconsistent

10. Geometric Interpretation

$a_1x + b_1y + c_1z = d_1$ .... (i)

$a_2x + b_2y + c_2z = d_2$ .... (ii)

$a_3x + b_3y + c_3z = d_3$ .... (iii)



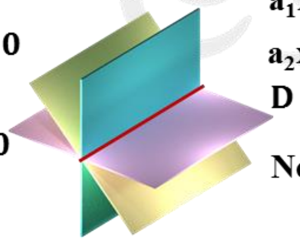
$D \neq 0$ Atleast any one of $D_x, D_y, D_z \neq 0$

Non - Trivial solution

$D \neq 0$ $D_x = D_y = D_z = 0$ Trivial solution

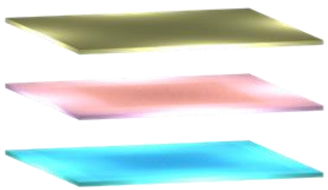
$D = 0$ $D_x = D_y = D_z = 0$

Infinite Sols.



$D = 0$ $D_x = D_y = D_z = 0$

No Solution



$d_1 = d_2 = d_3$

$a_1x + b_1y + c_1z = d_1$ .... (i)

$a_1x + b_1y + c_1z = d_2$ .... (ii)

$a_1x + b_1y + c_1z = d_3$ .... (iii)

$D = 0$ $D_x = D_y = D_z = 0$

Infinite Solution

$D = 0$ $D_x = D_y = D_z = 0$

No Solution

11. Homogeneous System

$a_1x + b_1y + c_1z = d_1$ .... (i)

$a_1x + b_1y + c_1z = d_2$ .... (ii)

$a_2x + b_2y + c_2z = d_3$ .... (iii)

$D = 0$ $D_x \neq 0$, or $D_y \neq 0$, or $D_z \neq 0$

No Solution

1. $D_x = D_y = D_z = 0$

2. Atleast one solution in this case is :
   $x = 0, y = 0, z = 0$

   known as Trivial solution

   $a_1x + b_1y + c_1z = 0$ .... (i)

   $a_2x + b_2y + c_2z = 0$ .... (ii)

   $a_3x + b_3y + c_3z = 0$ .... (iii)

3. System is always Consistent

4. If Sol. exist other than ($x = 0, y = 0, z = 0$), then that system has Non – Zero Sol. or Non – Trivial Sol.

D

$D_x = D_y = D_z = 0$

$\boxed{D \neq 0}$

1. Unique Solution
   $(x = 0, y = 0, z = 0)$
2. Trivial Solution
   Zero Solution

$\boxed{D = 0}$

1. ∞ No. of Solutions
2. Non – Trivial Solution
   Non – Zero Solution