MIND MAP

Definite Integration

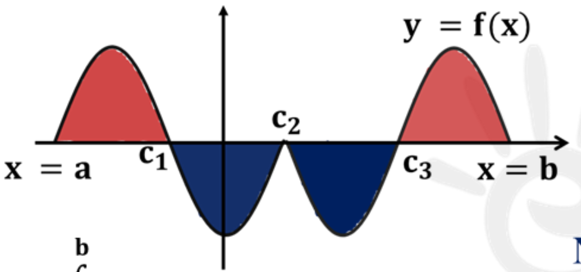
1. Geometrical Interpretation of Definite Integral

Introduction

$I = \int_a^b f(x)dx$

Represents algebraic sum of the areas of the figure bounded by :

- Curve $y = f(x)$
- $x -$ axis
- Lines $x = a$ and $x = b$



$I = \int_a^b f(x)dx$

Areas above $x -$ axis enter into this sum with (+) sign

below x–axis enter with (–) sign

If,

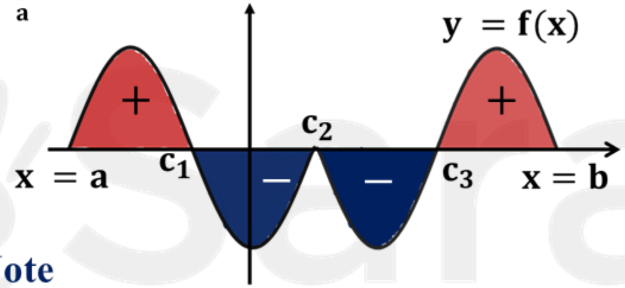
$\int_a^{c_1} f(x)dx = A_1 > 0$

$\int_{c_1}^{c_2} f(x)dx = A_2 < 0$

$\int_{c_2}^{c_3} f(x)dx = A_3 < 0$

$\int_{c_3}^{b} f(x)dx = A_4 > 0$

$\int_a^b f(x)dx = A_1 + A_2 + A_3 + A_4$



Note

1. If $f(x) > 0 \ \forall\ x \in [a, b]$ then,

$\int_a^b f(x)dx$ is always > 0 (when $a < b$)

If $f(x) < 0 \ \forall\ x \in [a, b]$ then,

$\int_a^b f(x)dx$ is always < 0 (when $a < b$)

2. If $\int_a^b f(x)dx = 0$

then the equation $f(x) = 0$ has at least one root lying in (a, b) provided it is a continuous function in (a, b) while the converse is not true.

2. Evaluating Definite Integrals by Finding Antiderivatives

Integration Using Substitution

$I = \int_{\ell_1}^{\ell_2} f(x)dx$

Make substitutions at $f(x)$, dx, ℓ_1, ℓ_2 and proceed further

3. Properties of definite integrals

❑ Property - 1

$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

(Change of variable does not change value of integral)

❑ Property - 2

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

MIND MAP

Definite Integration

Property - 3

$$\int_a^b f(x)dx = \int_a^c f(x)\,dx + \int_c^b f(x)dx$$

Note :

1. This property is useful when f(x) is not continuous in [a, b].
   Because we can break up the integral into several integrals at the points of discontinuity so that the function is continuous in the subinterval.

4. Walli's Theorem

1. $$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \frac{(n-1)(n-3)\ldots\ldots(1 \text{ or } 2)}{n(n-2)\ldots\ldots(1 \text{ or } 2)}K$$

$$\text{Where } K = \begin{cases} \frac{\pi}{2} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

5. Walli's Theorem

2. $$\int_0^{\pi/2} \sin^n x.\cos^m x\,dx$$

$$= \frac{[(n-1)(n-3)(n-5)\ldots.1 \text{ or } 2][(m-1)(m-3)\ldots.1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)\ldots.1 \text{ or } 2}K$$

$$\text{Where } K = \begin{cases} \frac{\pi}{2} & \text{if both } m \,\&\, n \text{ are even } (m, n \in N) \\ 1 & \text{otherwise} \end{cases}$$

Property - 4

$$\int_{-a}^{a} f(x)dx = \int_0^a [f(x) + f(-x)]dx$$

$$= \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)dx & \text{if } f(x) \text{ is even} \end{cases}$$

6. Properties of Definite Integrals

Property - 5

1. $$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

2. $$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

This property is known as King's Property .

7. Properties of Definite Integrals

Property - 6

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$$= \begin{cases} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \end{cases}$$

This property is also known as Queen's Property.

8. Properties of Definite Integrals

Property - 7

Let f(x) be periodic function with period 'T'

i.e. $f(T+x) = f(x)$

Then,

1. $$\int_0^{nT} f(x)dx = n\int_0^T f(x)\,dx \quad (n \in I)$$

2. $$\int_a^{T+a} f(t)dt = \int_0^T f(t)dt$$ and will be independent of a

MIND MAP

Definite Integration

Property - 7

3. $\int_{a}^{a+nT} f(x)dx = \int_{0}^{nT} f(x)dx = n\int_{0}^{T} f(x)dx$

4. $\int_{a+nT}^{b+nT} f(x)dx = \int_{a}^{b} f(x)\,dx$ where, $(n \in I)$

5. $\int_{mT}^{nT} f(x)dx = (n-m)\int_{0}^{T} f(x)\,dx$

where, $(n, m \in I)$

❑ Property - 8

Let $f^{-1}(x)$ is inverse function of $f(x)$, $x \in (a, b)$

$$\Rightarrow \int_{a}^{b} f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b) - af(a)$$

9. Derivative of Antiderivatives (Newton – Leibnitz Formula)

If h(x) & g(x) are differentiable functions of x and f(x) is a continuous function then,

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x)).h'(x) - f(g(x)).g'(x)$$

10. Sum of Series using Definite Integration

Introduction

$$\int_{a}^{b} f(x)dx = \lim_{\substack{h\to 0\\ n\to\infty}}\{h[f(a) + f(a+h) + f(a+2h) + \ldots\ldots + f(a+\overline{n-1}\,h)]\}$$

OR

$$\lim_{\substack{h\to 0\\ n\to\infty}} h\sum_{r=0}^{n-1} f(a+rh) = \int_{a}^{b} f(x)dx$$

Where, $(b - a = nh)$

If $a = 0$ & $b = 1$ then,

$$\lim_{\substack{h\to 0\\ n\to\infty}} h\sum_{r=0}^{n-1} f(rh) = \int_{0}^{1} f(x)dx$$

Where, $(nh = 1)$

Hence we have,

$$\int_{0}^{1} f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$$

11. Some Particular Cases

1. $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}f\left(\frac{r}{n}\right)$

OR

$$\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}f\left(\frac{r}{n}\right) = \int_{0}^{1} f(x)dx$$

2. $\lim_{n\to\infty}\sum_{r=1}^{pn}\frac{1}{n}f\left(\frac{r}{n}\right) = \int_{\alpha}^{\beta} f(x)dx$

Where, $\alpha = \lim_{n\to\infty}\frac{r}{n} = 0$ (as $r = 1$) $\quad \beta = \lim_{n\to\infty}\frac{r}{n} = p$ (as $r = pn$)