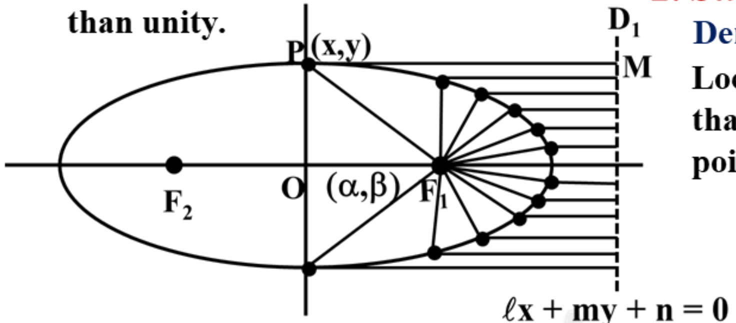
MIND MAP

Ellipse

1. General Equation of an Ellipse

Locus of a point P moving in such a way that ratio of its distance from a fixed point F (Focus) and fixed line (Directrix) is always constant and less than unity.



$\frac{PF_1}{PM} = e$ where, $0 < e < 1$

Focus : (α, β)

Directrix : $\ell x + my + n = 0$

$(PF_1)^2 = e^2(PM)^2$

$$\boxed{(x-\alpha)^2 + (y-\beta)^2 = e^2\frac{(\ell x + my + n)^2}{\ell^2 + m^2}}$$

This simplifies to

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

in which $\Delta \neq 0$, $h^2 < ab$

2. Standard Equation of Ellipse

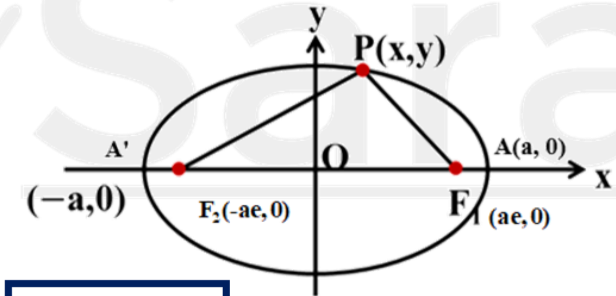
Derived using property:

Locus of a point moving in such a way that sum of its distances from two fixed points (called Focus) is always constant.

O : Centre F_1, F_2 : Focus

i.e. $|PF_1| + |PF_2| = 2a$

$|PF_1| + |PF_2| > F_1F_2$

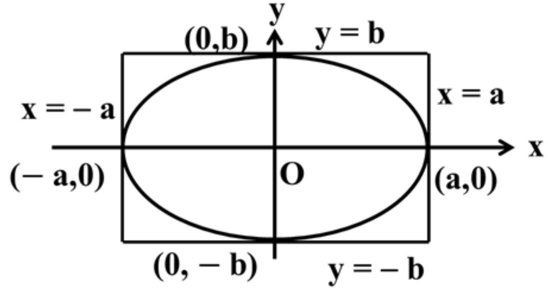


$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$, where $b^2 = a^2(1 - e^2)$

Note: In Standard Equation $(a > b)$

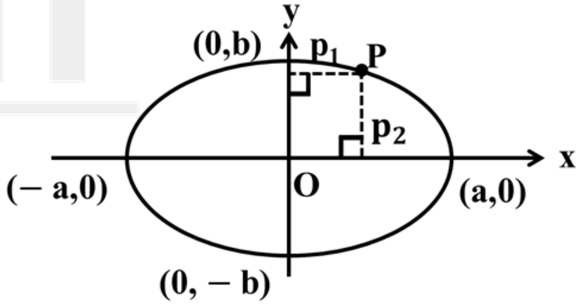
(i) It is a 2nd degree equation with powers of x and y both even and hence is symmetric about both the x and y axis.

(ii) The entire curve is confined within the rectangle bounded by the lines $x = \pm a$ and $y = \pm b$.



(iii) if p_1 and p_2 are the lengths of the perpendicular from any point P on the ellipse to minor and major axis then

$$\frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} = 1$$



3. Eccentricity

$$e = \frac{\text{distance from centre to focus}}{\text{distance from centre to vertex}}$$

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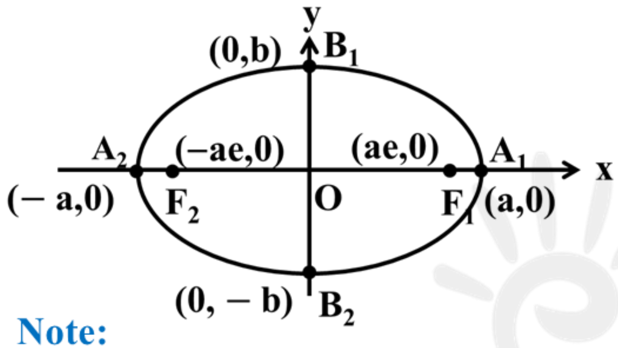
3. Eccentricity

$\boxed{e^2 = 1 - \frac{b^2}{a^2}}$ Hence, $e^2 = 1 - \left(\frac{m}{M}\right)^2$

m : length minor axis

M : length major axis

$\boxed{b^2 = a^2(1 - e^2)}$



Note:

(i) Two ellipses are said to be similar if they have the same value of eccentricity.

(ii) Degree of flatness of ellipse is defined as

e: $0 \Rightarrow b \to a \Rightarrow$ ellipse getting thicker & $\to$ circle

$1 \Rightarrow b \to 0 \Rightarrow$ ellipse getting thinner & $\to$ line segment between the two foci

Definitions & Basic Terminologies

Standard Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$

(I) Principal Axes

Major and minor axes together are known as Principal Axes of the ellipse.

(II) Centre

Point of intersection of the major and minor axis is called the Centre of the ellipse.

(III) Vertices

The points of intersection of the curve with focal axis are called the Vertices of the ellipse. $A_1(a,0)$ & $A_2(-a,0)$

(IV) Length of Major Axis

Distance between the two vertices i.e. 2a is the length of the Major Axis.

(V) Length of Minor Axis

The distance 2b i.e. B_1B_2 is the length of Minor Axis.

(VI) Focal Axis

The line containing the two fixed points (called focii) is called the focal axis.

(VII) Focal Length

The distance between F_1 & F_2 is called the Focal Length (= 2ae)

(VIII) Focal Chord

Any chord passing through focus is called a Focal Chord.

(IX) Double Ordinate

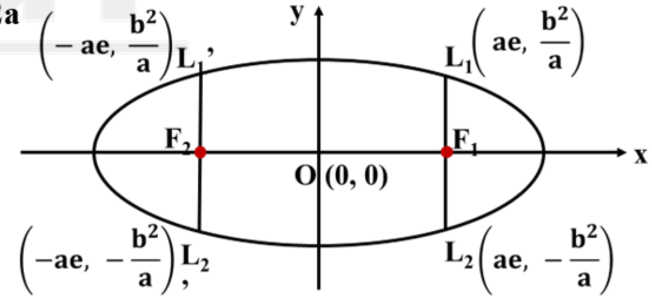
Any chord perpendicular to the focal axis is called Double Ordinate.

Latus Rectum

A particular double ordinate through focus or a particular focal chord perpendicular to focal axis is called its Latus Rectum.

Ends of Latus Rectum

$\boxed{\left(\pm ae, \pm \frac{b^2}{a}\right)}$



Length of Latus Rectum

$$\boxed{\text{Length of L.R.} = \frac{4b^2}{2a} = \frac{(\text{minor axis})^2}{\text{major axis}}}$$

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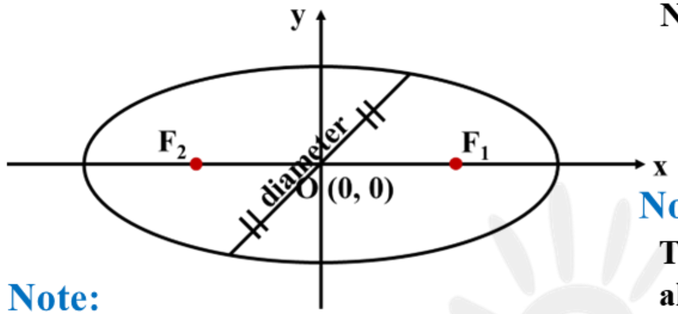
(XI) Foot of Directrix

The point of intersection of the focal axis with directrix is called the Foot of the Directrix.

(XII) Diameter

Any chord of the ellipse passing through the centre, gets bisected by it and is called the Diameter.



Note:

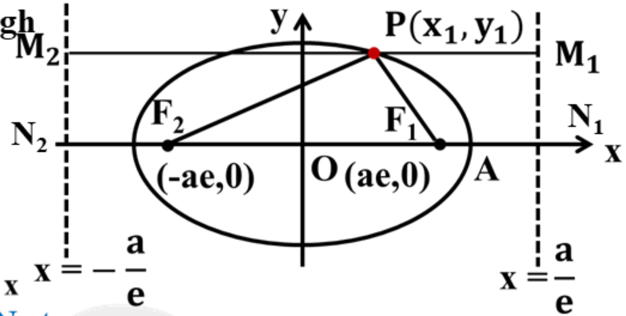
Distance of every focus from the extremity of minor axis is equal to 'a'.

$\Rightarrow F_1B_1 = a$

4. Directrix And Focal Directrix Property

(1) It is possible to define two lines, $x = \pm \frac{a}{e}$ corresponding to each focus, which satisfy the focal directrix property of the ellipse

i.e. $PF_1 = ePM_1$ & $PF_2 = ePM_2$

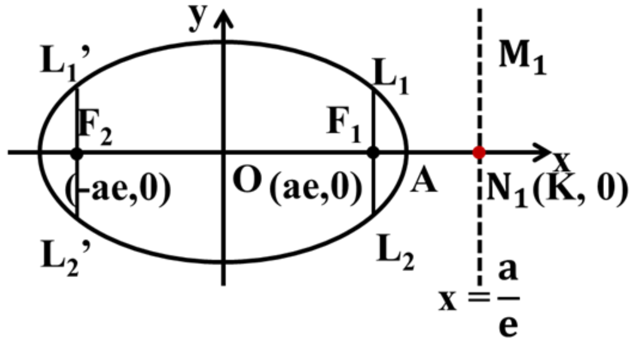


Note:

The length of the Latus Rectum can alternatively be expressed as

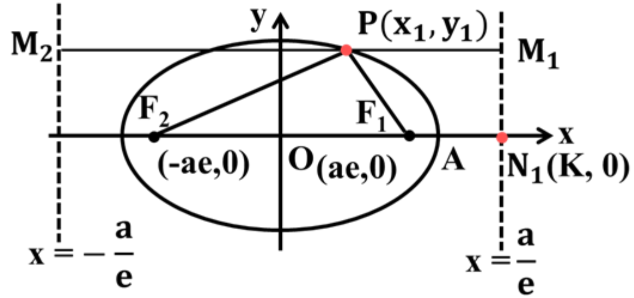
$L_1L_2 = 2e\,(F_1N_1)$

i.e. 2e times the distance between focus and corresponding foot of the directrix



(2) The sum of the focal radii of any point on the ellipse is equal to the length of the major axis. $PF_1 + PF_2 = 2a$

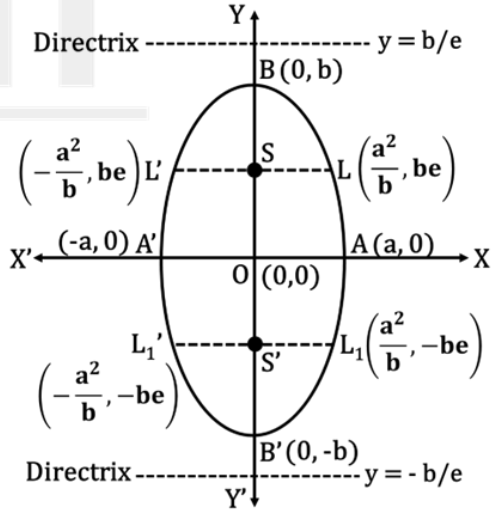
Note:



Ellipse With Vertical Major Axis

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a < b), a^2 = b^2(1 - e^2)$$

$AA' =$ Minor axis $= 2a$



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Ellipse

BB' = Major axis $= 2b$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \ (a<b)$

Eccentricity $e=\sqrt{1-\frac{a^2}{b^2}}$

$a^2=b^2(1-e^2)$

Focus $=(0,\pm be)$

Ends of the latus rectum

$L\left(\frac{a^2}{b},be\right), L'\left(-\frac{a^2}{b},be\right)$

$L_1\left(\frac{a^2}{b},-be\right), L_1'\left(-\frac{a^2}{b},-be\right)$

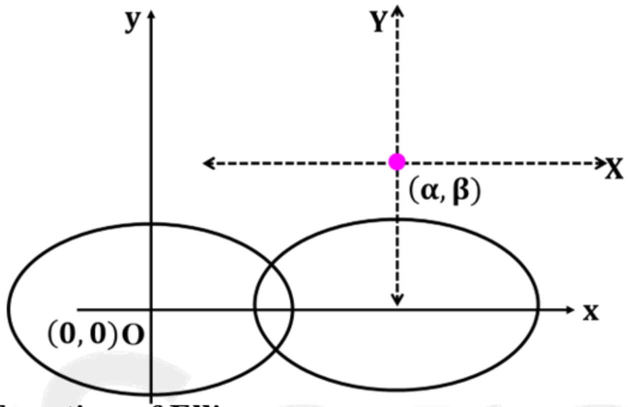
Length of LR = LL' = L_1L_1'

$=\sqrt{\left(\frac{a^2}{b}-\left(-\frac{a^2}{b}\right)\right)^2-(be-be)^2}=\frac{2a^2}{b}$

Equation of directrix $\quad y=\pm\frac{b}{e}$

5. Shifted Ellipse

If center is shifted to (α,β) and axes being parallel to coordinate axes & $a>b$



Equation of Ellipse:

$$\boxed{\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}=1}$$

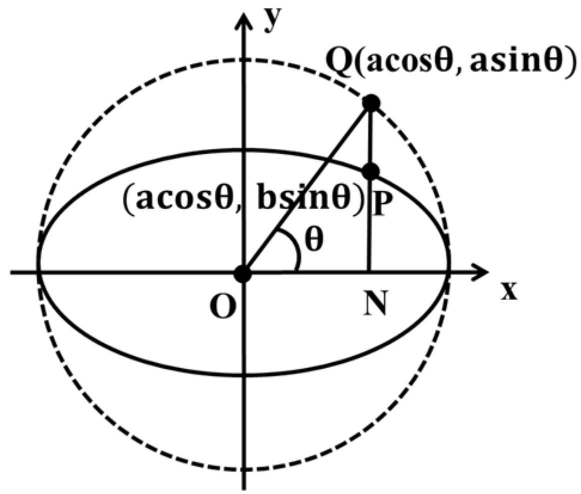
6. Auxiliary Circle And Eccentric Angle

Ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Circle described on the major axis of an ellipse as diameter is called the Auxiliary Circle.

Equation of the auxiliary circle is

$$\boxed{x^2+y^2=a^2}$$



$Q \equiv a\cos\theta, a\sin\theta$

$P \equiv (a\cos\theta,\ b\sin\theta)\ 0 \le \theta < 2\pi.$

P and Q are called corresponding points.

θ is called the eccentric angle of the point P.

Note:

1. $P(a\cos\theta, b\sin\theta)$ are called the parametric coordinates of a point on the ellipse.
2. Parametric Equation of Ellipse :

$$\boxed{x = a\cos\theta \ \& \ y = b\sin\theta}$$

3. $\frac{PN}{PQ}=\frac{b\sin\theta}{a\sin\theta-b\sin\theta}=\frac{b}{a-b}=$ constant

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7. Equation of Chord In Parametric Form

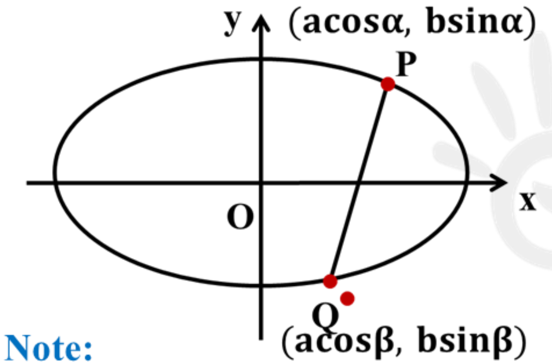
Ellipse : $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Points :

$P(\alpha) \equiv (a\cos\alpha,\ b\sin\alpha)$

$Q(\beta) \equiv (a\cos\beta,\ b\sin\beta)$

$$\boxed{\Rightarrow \frac{x}{a}\cos\frac{(\alpha+\beta)}{2}+\frac{y}{b}\sin\frac{(\alpha+\beta)}{2}=\cos\frac{(\alpha-\beta)}{2}}$$



Note:

For chord passing through (d, 0)

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{d-a}{d+a}$$

& For focal chord passing through (ae, 0)

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1} \quad \ldots(1)$$

Now,

For focal chord passing through (-ae, 0)

$$\tan\frac{\gamma}{2}\tan\frac{\delta}{2}==\frac{e+1}{e-1} \quad \ldots(2)$$

Multiplying (1) &(2) we get,

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=1$$

$$\boxed{\therefore \prod_{\alpha_i=\alpha}^{\delta}\tan\frac{\alpha_i}{2}=1}$$

8. Position of A Point w.r.t. Ellipse

Let $P \equiv (x_1, y_1)$

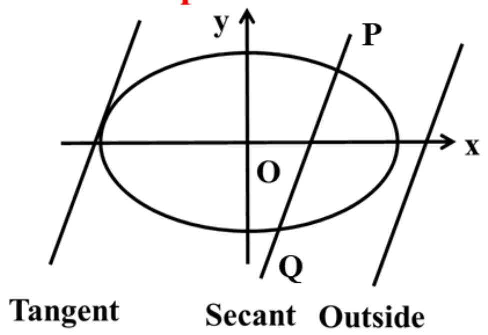
$S \equiv \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1 \qquad S_1 \equiv \dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1$

Point P lies on the Ellipse. If $S_1 = 0$

Point P lies inside the Ellipse. If $S_1 < 0$

Point P lies outside the Ellipse. If $S_1 > 0$

9. Line and Ellipse



Ellipse : $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \quad \ldots.(1)$

Line : $y = mx + c \quad \ldots.(2)$

solving (1) and (2)

$b^2x^2 + a^2(mx + c)^2 = a^2b^2$

$\Rightarrow (a^2m^2 + b^2)x^2 + 2a^2cmx + a^2(c^2 - b^2) = 0$

$\Rightarrow D = 4a^4m^2c^2 - 4a^2(c^2 - b^2)(b^2 + a^2m^2)$

$\Rightarrow D = 4a^2m^2(a^2m^2 + b^2 - c^2)$

- $D > 0$ Secant
- $D = 0$ Tangent $\Downarrow$ $c^2 = a^2m^2 + b^2$
- $D < 0$ Neither intersect nor touch

Length of Chord PQ $= \dfrac{2ab\left(\sqrt{a^2m^2+b^2-c^2}\right)(1+m^2)}{a^2m^2+b^2}$

Condition of Tangency : $D = 0$

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10. Tangent to Ellipse

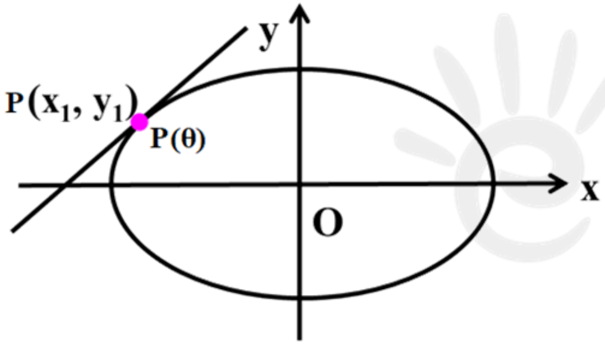
(i) Cartesian Form

Ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

Point : (x_1, y_1)

Equation of Tangent ; T = 0

$$\boxed{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0}$$

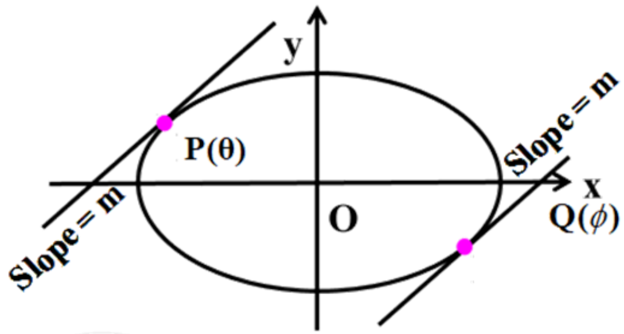


(ii) Parametric Form

Ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Point : $(a.\cos\theta, b.\sin\theta)$

$$\boxed{\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1}$$



The eccentric angles of point of contact of two parallel tangents differ by π.

(iii) Slope Form

$\because c^2 = a^2m^2 + b^2$ (By D = 0)

Hence, equation of tangent

$$\boxed{y = mx \pm \sqrt{a^2m^2+b^2}} \quad (m \in R)$$

i. e. $|\theta - \phi| = \pi$

Note:

(I) There are two parallel tangents for a given value of m.

(II) Point of intersection of the tangents at the points α &β

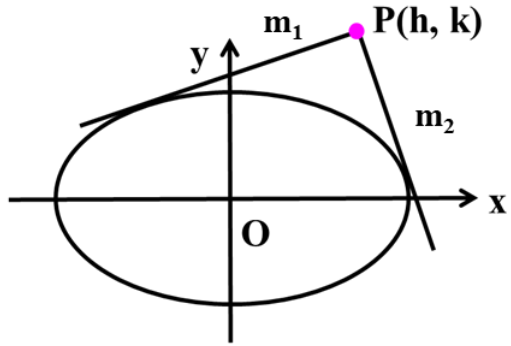
$$\boxed{P \equiv \left(a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)}$$

11. Tangents from external point

Passing through a given point there can be maximum of two tangents.

$(h^2-a^2)m^2 - 2hkm + k^2 - b^2 = 0$ → m_1, m_2

Also: $\boxed{\begin{aligned} m_1 + m_2 &= \frac{2kh}{h^2-a^2} \\ \& \ m_1 m_2 &= \frac{k^2-b^2}{h^2-a^2}\end{aligned}}$



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Ellipse

12. Director Circle

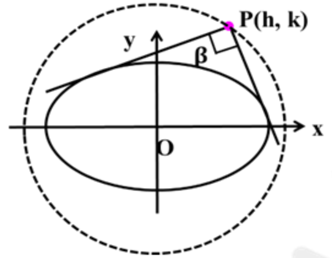
Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Tangents : $y = mx \pm \sqrt{a^2m^2 + b^2}$

if $\beta = 90°$ then,

$$\boxed{x^2 + y^2 = a^2 + b^2}$$

Which is the eq. of the Director Circle

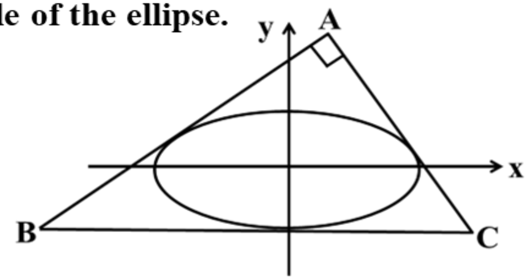


Hence Director Circle of an ellipse is a circle whose center is the center of ellipse and whose radius is $\sqrt{a^2 + b^2}$

Note:

If a right triangle ABC, right angled at A circumscribes an ellipse,

Then, locus of the point A is the director circle of the ellipse.

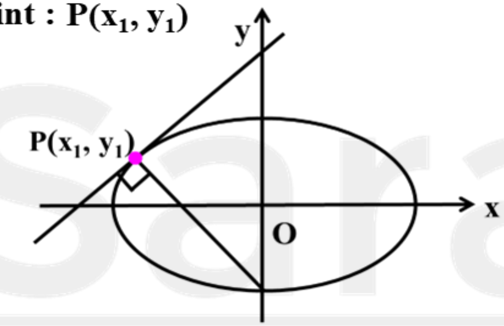


13. Normals to Ellipse

(i) Cartesian Form / Point Form

Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Point : $P(x_1, y_1)$



Eq. of Normal at point P :

$$\boxed{\Rightarrow \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2}$$

(ii) Parametric Form

Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Point : $P(a\cos\theta, b\sin\theta)$

Equation of Normal at point P :

$$\boxed{ax\sec\theta - by\operatorname{cosec}\theta = (a^2 - b^2)}$$

(iii) Slope Form

$$y = mx \pm \left(\frac{(a^2 - b^2)m}{\sqrt{a^2 + m^2b^2}}\right)$$

14. Pair of tangents & Chord of Contact

Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

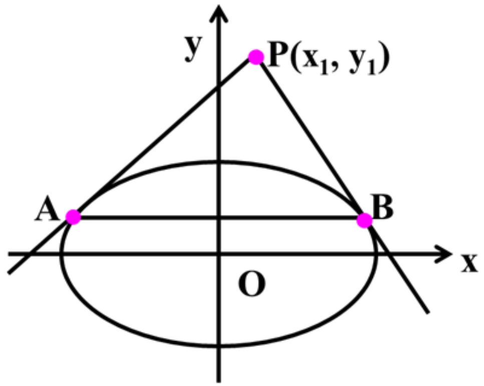
Point : $P(x_1, y_1)$ (Outside)

$$\boxed{\text{Pair of tangents: } SS_1 = T^2}$$

Where,

- $S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$
- $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$
- $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$

Pair of tangents



Chord of contact AB: T = 0

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

15. Chord with given Mid Point

Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Mid Point: (h, k)

Equation of Chord with mid point (h, k)

$$\boxed{T = S_1}$$

Where,

$$T = \frac{xh}{a^2} + \frac{yk}{b^2} - 1$$

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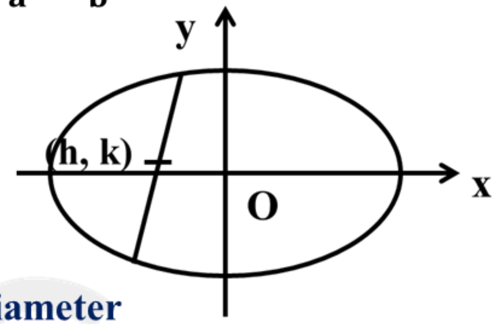
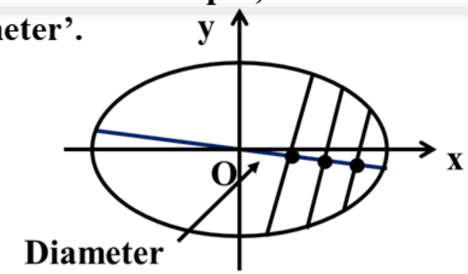
$$S_1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$$



16. Diameter

The locus of the middle points of a system of parallel chords with slope 'm' of an ellipse is a straight line passing through the center of the ellipse, called its 'Diameter'.



Equation of diameter :

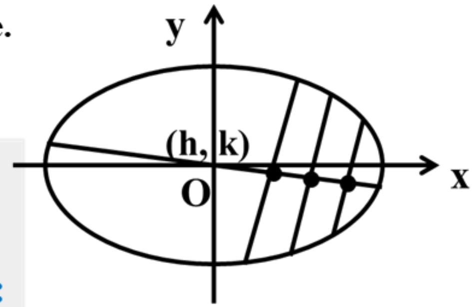
Equation of Chord with mid point (h, k)

$T = S_1$

$$\Rightarrow \frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$$

$$\Rightarrow m = -\frac{h}{a^2} \cdot \frac{b^2}{k};$$

Hence $\boxed{y = -\frac{b^2}{a^2 m} x}$ is the diameter of ellipse.



Note:

The tangents at the extremities of system of parallel chords intersect on the extended Diameter.