MIND MAP

Functions

1. Classification of Functions

(i) Polynomial Function

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$

Where n is a non-negative integer & $a_0, a_1, a_2, \ldots, a_n$ are real numbers & $a_0 \neq 0$.
A polynomial function is always continuous.

Note

(A) A polynomial of degree one with no constant term is called an odd linear function i.e. $f(x)=a\,x$, $a_0 \neq 0$

(B) There are two polynomial functions, satisfying the relation

$f(x) \cdot f(1/x) = f(x) + f(1/x)$

(a) $f(x) = x^n + 1$ (b) $f(x) = 1 - x^n$

where n is a positive integer.

(C) A polynomial of degree odd has its range $(-\infty, \infty)$ put a polynomial of degree even has a range which is always subset of R.

(ii) Algebraic Function

A function f is called an algebraic function if it can be constructed using algebraic operations such as addition, subtraction, multiplication, division and taking roots, started with polynomials.

Note

All polynomial are algebraic but converse is not true. Functions which are not algebraic, are knows as Transcendental function.

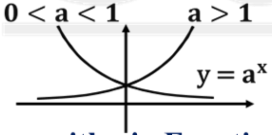
(iii) Fractional Rational Function

$$y = f(x) = \frac{g(x)}{h(x)}$$

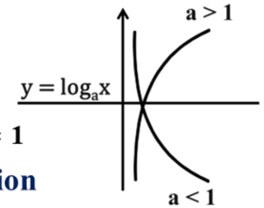
where g (x) & h (x) are polynomials & h (x) ≠ 0. The domain of f (x) is set of real x such that h (x) ≠ 0.

(iv) Exponential Function

$f(x) = a^x = e^{x\ln a}$ $(a > 0, a \neq 1, x \in R)$
$f(x) = a^x$ is called an exponential function because the variable x is the exponent.



(v) Logarithmic Function



$y = \log_a x$, $x > 0$, $a > 0$, $a \neq 1$

(vi) Absolute Value Function

$y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as-

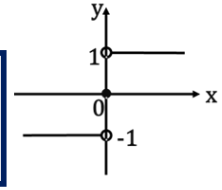
$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For $f(x) = |x|$, domain is R and range is $R^+ \cup \{0\}$

(vii) Signum Function

A function y= f(x) = Sgn (x) is defined as follows.

$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

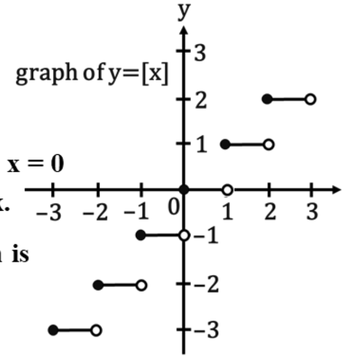


y = sgn x

$\text{Sgn } x = |x|/x$ or $\dfrac{x}{|x|}$ $x \neq 0$; $= 0$ x = 0

(viii) Greatest Integer or Step Up Function

It is also written as
$y = f(x) = [x]$
Where, [x] denotes the greatest integer less than or equal to x.



For $f(x) = [x]$, domain is R and range is I.

Properties of greatest integer function

(a) $x - 1 < [x] \leq x \Rightarrow 0 \leq x - [x] < 1$

(b) $[x+m] = [x] + m$, if m is an integer.

(c) $[x] + [-x] = \begin{cases} 0 & \text{if x is an integer} \\ -1 & \text{otherwise} \end{cases}$

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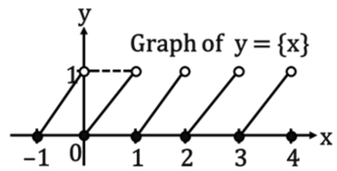
Functions

(ix) Fractional Part Function

It is defined as : $f(x) = \{x\} = x - [x]$.

The period of this function is 1

Domain is R and range is [0, 1)



Properties of fractional part

(a) $0 \le \{x\} < 1$

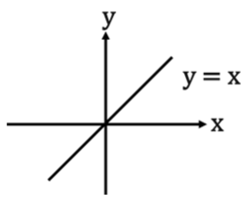
(b) $\{x + n\} = \{x\},\ n \in I$

(c) $\{x\} + \{-x\} = \begin{cases} 0, & x \in I \\ 1, & x \notin I \end{cases}$

(x) Identity Function

$f : A \to A$ defined by $f(x) = x, \forall x \in A$ is called the identity of A and is denoted by I_A

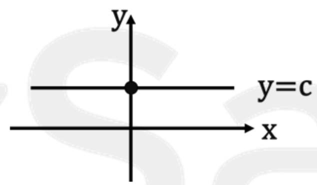
$\boxed{f : R \to R,\ \ f(x) = x}$



(xi) Constant Function

$f : A \to B;\ f(x) = c,\ \forall\ x \in A,\ c \in B$ is a constant function.

Note that the range of a constant function is a singleton



2. Range

Range of $y = f(x)$ is the collection of all outputs corresponding to each real number of the domain.

3. Equal or Identical Function

Two functions f & g are said to be equal if

(i) The domain of f = the domain of g.
(ii) The range of f = the range of g and
(iii) $f(x) = g(x)$, for every x belonging to their common domain.

Note Functions are also equal if their graphs are same

4. One-One Function

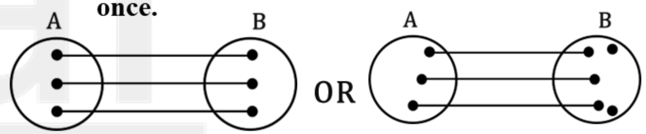
(Injective mapping)

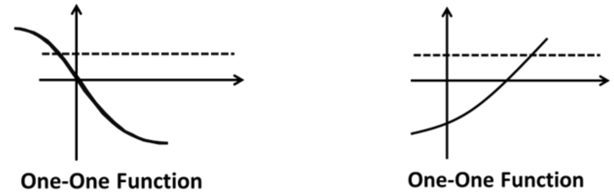
A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$,

If $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \ne x_2 \Leftrightarrow f(x_1) \ne f(x_2)$.

Note (i) A continuous function which is always increasing or decreasing in whole domain, then $f(x)$ is one-one.

(ii) A function is one to one if and only if a horizontal line intersects its graph at most once.





5. Many-one function (not injective)

$f : A \to B$ is many one if for; $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \ne x_2$.

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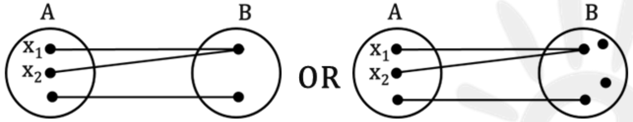
Functions

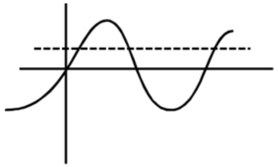
Note

(i) Any continuous function which has atleast one local maximum or local minimum in its domain, then f(x) is many-one.

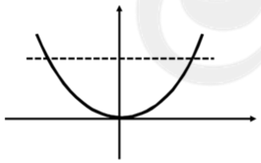
In other words, if a line parallel to x-axis cuts the graph of the function atleast at two points, then f is many-one.

(ii) If a function is one-one, it cannot be many-one and vice versa. One One + Many One = Total number of mappings.





Many-One Function

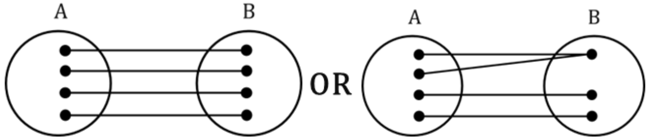


Many-One Function

Note

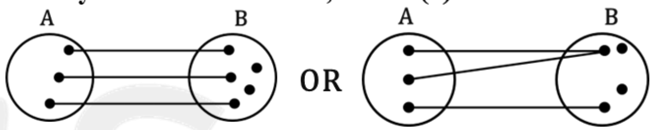
If range = co-domain, then f (x) is onto. Any polynomial of degree odd , $f : R \to R$ is onto.

Surjective mapping can be shown as



6. Into function

If $f : A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f(x) is into.



Note

(i) If a function is onto, it cannot be into and vice versa . A polynomial of degree even define from $R \to R$ will always be into & a polynomial of degree odd defined from $R \to R$ will always be onto.

(ii) If f is both injective & surjective, then it is called a Bijective mapping.

Some important points to remember

If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n, \; n \in R$

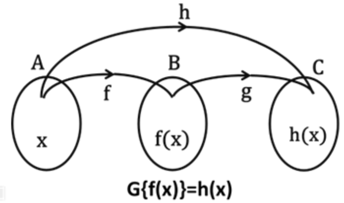
(iii) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx} \Rightarrow f(x) = A^x$, $A > 0$.

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

7. Composite Function

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function $gof : A \to C$ defined by $(gof)(x) = g(f(x)) \; \forall x \in A$

$$\xrightarrow{x} [f] \xrightarrow{f(x)} [g] \to g(f(x))$$



Note

gof in general not equal to fog.

8. Properties of Composite Functions

(i) $gof \neq fog$.

(ii) If f, g, h are three functions such that fo (goh) & (fog) oh are defined, then fo (goh) = (fog) oh.

Associativety : $f : (N) \to I_0 \quad f(x) = 2x$

$g: I_0 \to Q \quad g(x) = \frac{1}{x} \quad h: Q \to R \quad h(x) = e^{\frac{1}{x}}$

$(hog)of = ho(gof) = e^{2x}$

(iii) The composite of two bijections is a bijection i.e. if f and g are two bijections such that gof is defined, then gof is also a bijection.

9. Homogeneous Functions

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

Note

f (x, y) is a homogeneous function iff $f(tx, ty) = t^n f(x, y)$

10. Bounded Function

A function is said to be bounded if $|f(x)| \le M$, where M is a finite quantity.

11. Implicit & Explicit Function

A function defined by an equation not solved for the dependent variable is called an **Implicit Function**.

If y has been expressed in terms of x alone then it is called an **Explicit Function**.

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Functions

12. Odd & Even Functions

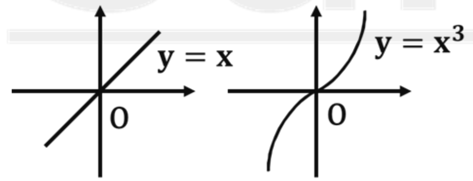
A function f (x) defined on the symmetric interval (–a, a)

If f (-x) = f (x) for all x in the domain of 'f' then f is said to be an **even function**.

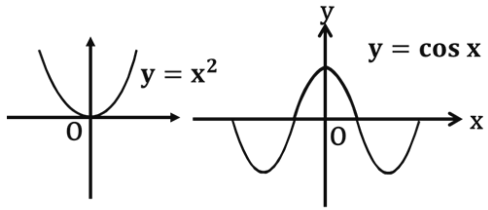
If f (-x) = -f (x) for all x in the domain of 'f' then f is said to be an **odd function**.

Note

Odd functions (Symmetric about origin)



Even functions (Symmetric about y-axis)



(a) f (x) - f (-x) = 0 => f (x) is even & f (x) + f (-x) = 0 => f (x) is odd .

(b) A function may neither be odd nor even .

(c) Inverse of an even function is not defined and an even function can not be strictly monotonic

(d) Every function can be expressed as the sum of an even & an odd function.

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{ODD}}$$

(f) The only function which is defined on the entire number line & is even and odd at the same time is f(x) =0. Any non zero constant is even.

(g) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd then f.g will be odd .

f(x)	g(x)	f(x) + g(x)	f(x) − g(x)	f(x).g(x)	f(x)/g(x)	gof(x)	fog(x)
Odd	Odd	Odd	Odd	Even	Even	Odd	Odd
Even	Even	Even	Even	Even	Even	Even	Even
Odd	Even	NENO	NENO	Odd	Odd	Even	Even
Even	Odd	NENO	NENO	Odd	Odd	Even	Even

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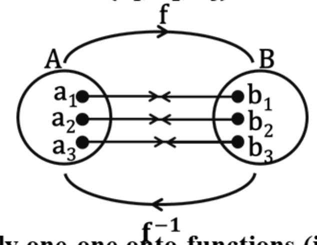
13. Inverse of A Function

Let $f: A \to B$ be a one-one & onto function, then there exists a unique function $g: B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x$, $\forall\ x \in A$ & $y \in B$. Then g is said to be inverse of f.
Thus $g = f^{-1}: B \to A = \{(f(x), x)\ \forall\ (x, f(x)) \in f\}$.

Domain of $f = \{a_1, a_2, a_3\}$ = Range of f^{-1}
Range of $f = \{b_1, b_2, b_3\}$ = Domain of f^{-1}



Note

(a) Only one-one onto functions (i.e., Bijections) are invertible.

(b) To find the inverse
Step-1: write $y = f(x)$
Step-2: solve this equation for x in terms of y (if possible)
Step-3: To express f^{-1} as a function of x, interchange x and y.

14. Properties of inverse of a function

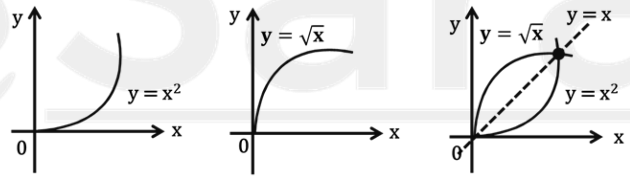
(i) The inverse of Bijection is unique.

(ii) The inverse of Bijection is also bijection.

(iii) If $f: A \to B$ is Bijection & $g: B \to A$ is inverse of f, then $fog = I_B$ & $gof = I_A$, where I_A, I_B are the identical function on the set A and B respectively

(iv) If $f: A \to B$ and $g: B \to C$ are two bijections, then $gof: A \to C$ is bijections and $(gof)^{-1} = f^{-1}og^{-1}$.

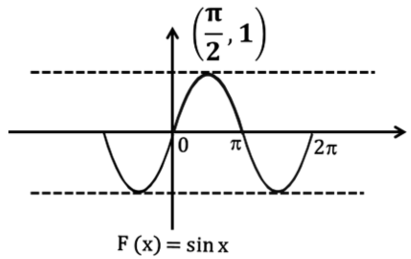
(v) The graphs of f & g are the mirror images of each other in the line $y = x$.



15. Periodic Function

A function f(x) is called periodic if there exists a positive number $T(T > 0)$ called the period of the function such that $f(x + T) = f(x)$, for all values of x within the domain of x.

If the graph repeats at fixed interval then function is said to be periodic and its period is the width of that interval.



16. Properties of periodic function

(i) $f(T) = f(0) = f(-T)$, where 'T' is the period.

(ii) Inverse of a periodic function does not exist.

(iii) Every constant function is always periodic, with no fundamental period.

(iv) If f(x) has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(v) if f(x) has a period T then $f(ax + b)$ has a period $\frac{T}{|a|}$.

(vi) If f(x) has a period T & g(x) also has a period T then it does not mean that $f(x) + g(x)$ must have a period T.
e.g. $f(x) = |\sin x| + |\cos x|$; $\sin^4 x + \cos^4 x$ has fundamental period equal to $\frac{\pi}{2}$.

(vii) If f(x) and g(x) are periodic then $f(x) + g(x)$ need not be periodic.