MIND MAP

Indefinite Integration

1. Elementary Integrals

If f and g are functions of x such that

$$\frac{d}{dx}(g(x)+C)=f(x) \Rightarrow \int f(x)dx = g(x)+C$$

2. Properties of Integration

1. $\int (f_1(x) \pm f_2(x))dx = \int f_1(x)dx \pm \int f_2(x)dx$

2. $\int af(x)dx = a\int f(x)dx$

3. If, $\int f(x)dx = g(x)+C$

$\Rightarrow \int f(ax+b)dx = \frac{1}{a}g(ax+b)+C$

4. $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

5. $\int\left(\frac{d}{dx}f(x)\right)dx = f(x)+C$

3. Important Formulas

1. (a) $\int x^n dx = \frac{x^{n+1}}{n+1}+C \ (n \neq -1, n \in R)$

(b) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}+C$

$(n \neq -1, n \in R)$

2. (a) $\int \frac{1}{x}dx = \ell n|x|+C$

(b) $\int \frac{dx}{ax+b} = \frac{\ell n|ax+b|}{a}+C$

3. (a) $\int e^x dx = e^x + C$

(b) $\int a^x dx = \frac{a^x}{\ell na}+C \ (a>0,\ a\neq 1)$

$\left(\because \frac{d}{dx}(a^x) = a^x \ell na\right)$

(c) $\int e^{ax+b}dx = \frac{e^{ax+b}}{a}+C$

(d) $\int a^{px+q}dx = \frac{1}{p}\frac{a^{px+q}}{\ell na}+C \ (a>0,\ a\neq 1)$

4. (a) $\int \sin x\, dx = -\cos x + C$

(b) $\int \cos x dx = \sin x + C$

(c) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+C$

(d) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+C$

5. (a) $\int \sec^2 x\, dx = \tan x + C$

(b) $\int \mathrm{cosec}^2 x\, dx = -\cot x + C$

(c) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)+C$

(d) $\int \mathrm{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)+C$

6. (a) $\int \sec x \tan x\, dx = \sec x + C$

(b) $\int \mathrm{cosec}\, x \ \cot x\, dx = -\mathrm{cosec}\, x + C$

(c) $\int \sec(ax+b).\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+C$

(d) $\int \mathrm{cosec}(ax+b).\cot(ax+b)dx = -\frac{1}{a}\mathrm{cosec}(ax+b)+C$

7. (a) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + C$

(b) $\int -\frac{dx}{\sqrt{1-x^2}} = \cos^{-1}x + C$

(c) $\int \frac{dx}{1+x^2} = \tan^{-1}x + C$

(d) $\int -\frac{dx}{1+x^2} = \cot^{-1}x + C$

(e) $\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1}x + C$

(f) $\int -\frac{dx}{|x|\sqrt{x^2-1}} = \mathrm{cosec}^{-1}x + C$

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4. Integration by Substitution

If, $\int f(x)dx = \phi(x)$ $\int f(g(x))g'(x)dx = ?$ ... (i)

Let, $g(x) = t \Rightarrow g'(x)dx = dt$

Eq. (i) becomes, $\int f(t)dt = \phi(t) + C$

NOTE: Substitution is appropriate if the integrand in (i) is a loving one.

Direct Substitution

$I = \int [f(x)]^n f'(x)\,dx$ or $\int \frac{f'(x)}{[f(x)]^n}dx$

Put, $f(x) = t \Rightarrow f'(x)\,dx = dt$

$\Rightarrow I = \int t^n\,dt = \frac{t^{n+1}}{n+1} + C$

$= \frac{1}{n+1}[f(x)]^{n+1} + C$

Special Case

For, $I = \int \frac{f'(x)}{f(x)}dx = \ell n|f(x)| + C$

(1) $\int \cot x dx = \ell n\,|\sin x| + C$

(2) $\int \tan x dx = \ell n\,|\sec x| + C$

(3) $\int \sec x dx = \ell n\,|\sec x + \tan x| + C$

or $\ell n\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

(4) $\int \operatorname{cosec} x dx = \ell n\,|\cot x - \operatorname{cosec} x| + C$

or $\ell n\left|\tan\frac{x}{2}\right| + C$

Standard Substitution

1. (a) $x^2 + a^2$ or $\sqrt{x^2 + a^2}$

put $x = a\tan\theta$ or $a\cot\theta$

(b) $x^2 - a^2$ or $\sqrt{x^2 - a^2}$

put $x = a\sec\theta$ or $a\operatorname{cosec}\theta$

(c) $a^2 - x^2$ or $\sqrt{a^2 - x^2}$

put $x = a\sin\theta$ or $a\cos\theta$

2. If both $\sqrt{a + x}$, $\sqrt{a - x}$ are present, then

put $x = a\cos 2\theta$

3. (a) $\sqrt{(x - a)(b - x)}$ or $\sqrt{\frac{x - a}{b - x}}$

put $x = a\cos^2\theta + b\sin^2\theta$

(b) $\sqrt{(x - a)(x - b)}$ or $\sqrt{\frac{x - a}{x - b}}$

put $x = a\sec^2\theta - b\tan^2\theta$

Some Standard Formulas

1. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

2. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

3. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ell n\left[x + \sqrt{x^2 + a^2}\right] + C$

4. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ell n\left[x + \sqrt{x^2 - a^2}\right] + C$

5. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ell n\left|\frac{x + a}{x - a}\right| + C$

6. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ell n\left|\frac{x - a}{x + a}\right| + C$

5. Integration By Parts

Let, $I = \int \underset{\mathrm{I}}{f(x)}.\underset{\mathrm{II}}{g(x)}\,dx$

$= f(x)\cdot\int g(x)dx - \int\left\{f'(x)\left(\int g(x)dx\right)\right\}dx$

$= \mathrm{I}.\int \mathrm{II}\,dx - \int\left\{\frac{d}{dx}(\mathrm{I})\left(\int(\mathrm{II})dx\right)\right\}dx$

Move left → right for I & II

I L A T E

- I → Inverse
- L → Logarithmic
- A → Algebraic
- T → Trigonometric
- E → Exponential

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Indefinite Integration

5. Integration By Parts

Some Important Formulas

1. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\ell n\left(x+\sqrt{x^2+a^2}\right)+C$

2. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ell n\left(x+\sqrt{x^2-a^2}\right)+C$

3. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

Two Classic Integrands

1. $\int (f(x)+x.f'(x))\,dx = x\,f(x)+C$

2. $\int e^x(f(x)+f'(x))dx = e^x f(x)+C$

Integrals of the Type

1. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+C$

2. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+C$

6. Integration of Trigonometric Functions

TYPE – I $\int \frac{dx}{a+b\sin^2x}$ / $\int \frac{dx}{a+b\cos^2x}$ / $\int \frac{dx}{a\sin^2x+b\cos^2x+c\sin x\cos x+d}$

Divide N^r and D^r by $\sin^2x$ or $\cos^2x$

TYPE – II $\int \frac{1}{a\sin x+b\cos x+c}dx$ / $\int \frac{1}{a\sin x+b\cos x}dx$

OR $\int \frac{1}{a+b\sin x}dx$ / $\int \frac{1}{a+b\cos x}dx$

OR Multiply by $\sec^2x$

Then put $\tan x = t$ & $\sec^2x\,dx = dt$

TYPE – III $\int \frac{a\sin x+b\cos x+c}{\ell\sin x+m\cos x+n}dx$

TYPE – IV Put Nr. = A(Dr.) + B(Dr.)' + C

1. Den : $a+\sin 2x$ or $\sqrt{b+\sin 2x}$

Num : $\cos x - \sin x$ or $\cos x+\sin x$

2. Den : $\cos x+\sin x$ or $\cos x-\sin x$

Num : $a+\sin 2x$

Then put $\sin 2x = (\sin x+\cos x)^2-1$

or $\sin 2x = 1-(\sin x-\cos x)^2$

TYPE – V $I = \int \sin^m x.\cos^n x\,dx$

Case – I At least one of m & n odd integer

Put ($\sin x$ or $\cos x$) = t

Case – II Both $m, n \in$ Even Integers

$\cos^2x = \frac{1+\cos 2x}{2}$ $\sin^2x = \frac{1-\cos 2x}{2}$

Case – III If m + n is a negative integer

Put ($\tan x$) = t

7. Integration of Rational Functions

Rational Func $f(x) = \frac{\text{Polynomial } N(x)}{\text{Polynomial } D(x)}$

Split the rational functions into partial fractions according to the factors of D(x).

Case - I Degree [N(x)] < Degree [D(x)]

(A) When denominator D(x) has non-repeating linear factors.

$D(x) = (x-a_1)(x-a_2) \ldots\ldots (x-a_n)$

$\Rightarrow \frac{N(x)}{D(x)} = \frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\ldots\ldots+\frac{A_n}{x-a_n}$

where $A_1, A_2 \ldots\ldots A_n$ are constants to be find

Take LCM of RHS and equate the numerators of RHS & LHS

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(B) When denominator D(x) has some repeating linear factors.

$$D(x) = (x-a)^k (x-a_1)(x-a_2) \ldots (x-a_n)$$

$$\Rightarrow \frac{N(x)}{D(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_k}{(x-a)^k} + \frac{B_1}{x-a_1} + \ldots + \frac{B_n}{x-a_n}$$

7. Integration of Rational Functions

Case - I

Degree [N(x)] < Degree [D(x)]

(C) When denominator D(x) has some quad factors (which can not be factorised further) repeating or non – repeating.

$$\frac{N(x)}{D(x)} = \frac{Ax+B}{px^2+qx+r}$$

$$\frac{N(x)}{D(x)} = \frac{Ax+B}{px^2+qx+r} + \frac{Cx+D}{(px^2+qx+r)^2}$$

Case - II Degree [N(x)] ≥ Degree [D(x)]

$$\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

Degree [R(x)] < Degree [D(x)]

Now proceed, as earlier methods.

8. Integration of Algebraic Expressions

1. $\int \frac{px+q}{ax^2+bx+c}\,dx$

Write, $px+q = m\left(ax^2+bx+c\right)' + n$

$\Rightarrow px+q = m(2ax+b) + n$

Compare LHS & RHS to find (m & n)

Now put, $ax^2+bx+c = t$ & $(2ax+b)dx = dt$

2. $I = \int \frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}\,dx$

px^2+qx+r
$= A\left(ax^2+bx+c\right) + B\left(ax^2+bx+c\right)' + C$

Find A, B and C and proceed as earlier.

9. Integrand Involving Even Powers of x

$$\int \frac{x^2+1}{x^4+kx^2+1}\,dx \quad \Big| \quad \int \frac{x^2-1}{x^4+kx^2+1}\,dx$$

For k = 0, Integrals are termed as Algebraic Twins.

$$\int \frac{(x^2+1)}{x^4+1}\,dx \quad \Big| \quad \int \frac{(x^2-1)}{x^4+1}\,dx$$

10. Integration of Irrational Algebraic Functions

Type-1 $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ (Put, $px+q=t^2$)

Type-2 $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$ (Put, $px+q=t^2$)

Type-3 $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ $\left(\text{Put}, ax+b=\frac{1}{t}\right)$

Type-4 $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$ $\left(\text{Put}, x=\frac{1}{t}\right)$

Or the trigonometric substitutions are also helpful.