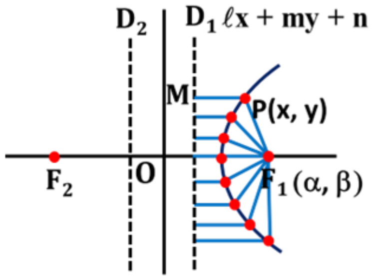
MIND MAP

Hyperbola

1. General Equation of Hyperbola



$$\frac{PF_1}{PM} = e$$

where, $e > 1$

Locus of a point P moving in such a way that ratio of its distance from a fixed point F (Focus) and fixed line (Directrix) is always constant.

Focus : (α, β)

Directrix : $\ell x + my + n = 0$

$(PF_1)^2 = e^2(PM)^2$

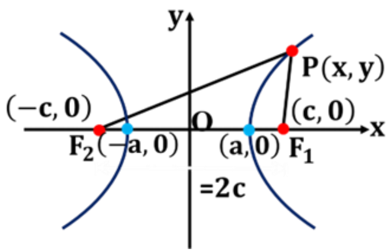
$$\boxed{(x-\alpha)^2 + (y-\beta)^2 = e^2\frac{(\ell x + my + n)^2}{\ell^2 + m^2}}$$

this simplifies to

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

in which $\Delta \neq 0$, $h^2 > ab$

2. Standard Equation of Hyperbola



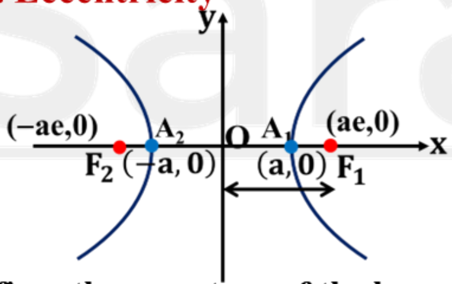
$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

Note :

- This is same as the standard eq. of ellipse except b^2 which has been replaced by '$-b^2$'.
- This is a 2nd degree eq. in which the powers of x and y are even, hence the curve is symmetric about both the axes.
- $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$

$\Rightarrow x \in (-\infty, -a] \cup [a, \infty)$

3. Eccentricity



Defines the curvature of the hyperbola

$$e = \frac{\text{Distance from centre to focus}}{\text{Distance from centre to vertex}}$$

$$= \frac{OF_1}{OA_1} = \frac{c}{a} \Rightarrow c = ae \ (e > 1)$$

Hence, the foci are (ae, 0) & (-ae, 0)

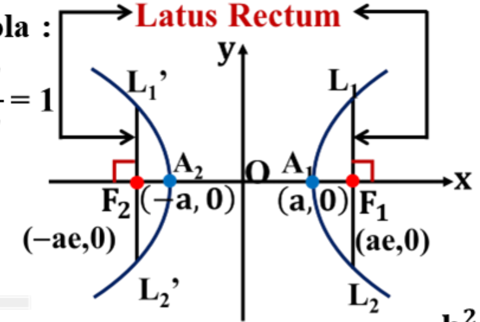
Remember:

- $a^2e^2 = a^2 + b^2$
- $b^2 = a^2(e^2 - 1)$
- Two hyperbolas are said to be similar if they have the same value of eccentricity.
- Equation of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of eccentricity can be written as

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

4. Latus Rectum

Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Substituting $x = ae$, $x = \pm ae$, $y = \pm\frac{b^2}{a}$

Coordinates of the ends of Latus Rectum

$$\boxed{\equiv \left(\pm ae, \ \pm\frac{b^2}{a}\right)}$$

Latus Rectum L_1L_2

$$= \frac{2b^2}{a} = 2a(e^2 - 1)$$

$$= \frac{(\text{Conjugate Axis})^2}{\text{Transverse Axis}}$$

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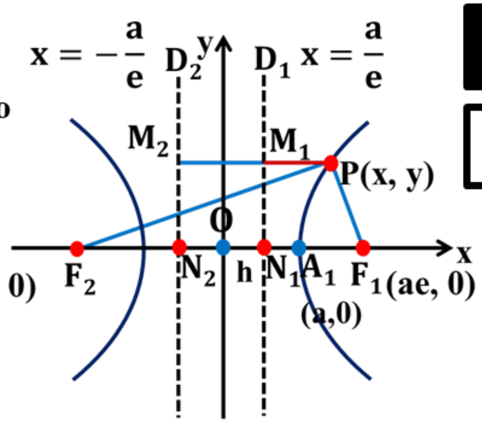
5. Directrix

Corresponding to each focus, there are 2 lines

$$\boxed{x = \pm \frac{a}{e}}$$

which are known as Directrices.



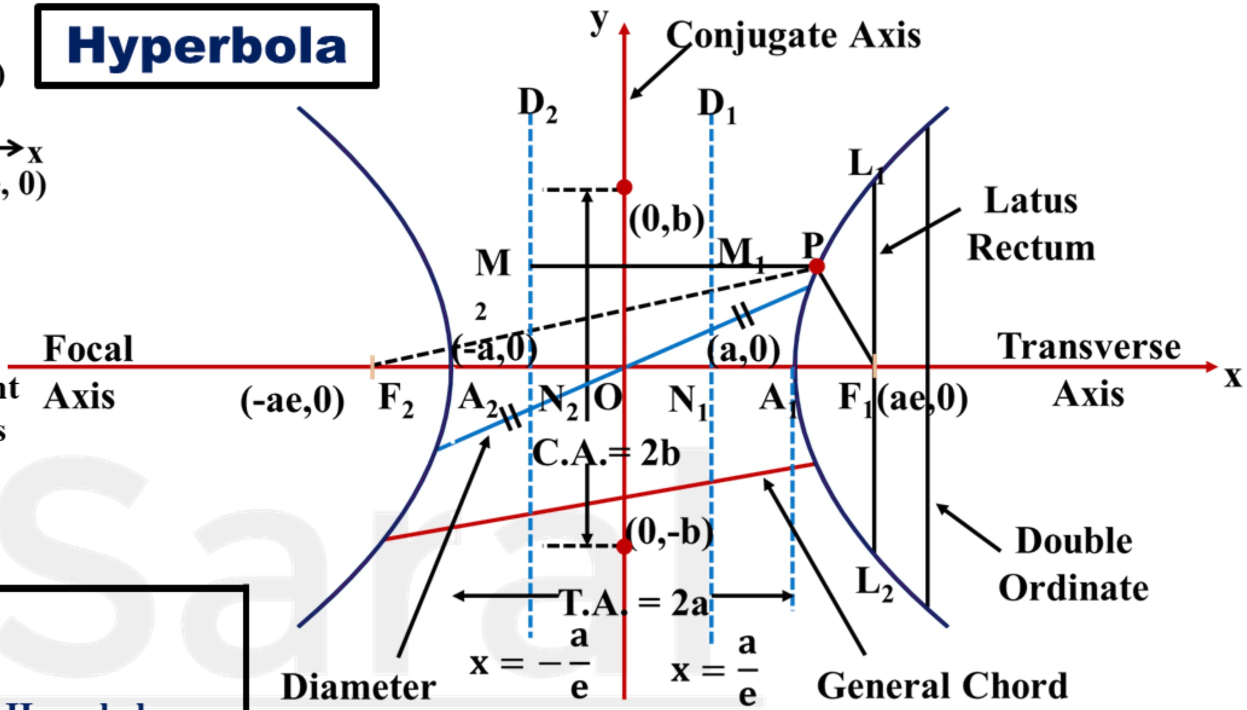
Note:

(i) The difference of the focal radii of any point on the hyperbola is equal to the length of its transverse axis. Hence, $|PF_1 - PF_2| = 2a$

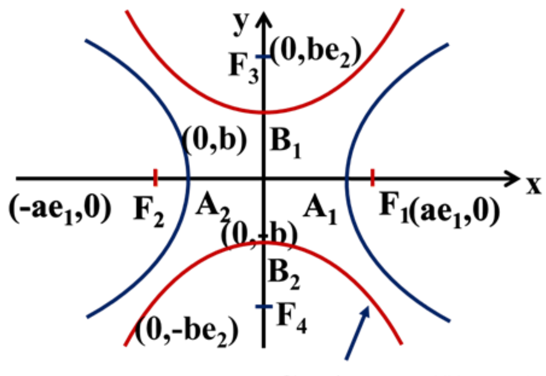
(ii) Foot of the directrix (N_1 and N_2).

$\boxed{(-a/e, 0)}$ $\boxed{(a/e, 0)}$

6. Hyperbola at a Glance



7. Conjugate Hyperbola



For the Hyperbola,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad ....(1)$$

with eccentricity : e_1

Conjugate Hyperbola is,

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \qquad ....(2)}$$

with eccentricity : e_2

Also,

$$e_1^2 = 1 + \frac{b^2}{a^2} = \frac{a^2 + b^2}{a^2}$$

$$e_2^2 = 1 + \frac{a^2}{b^2} = \frac{a^2 + b^2}{b^2}$$

And, $\boxed{\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1}$

Note : The focii of a hyperbola & its conjugate are concyclic & form the vertices of a square.

$\Rightarrow F_1F_3F_2F_4$ is a square

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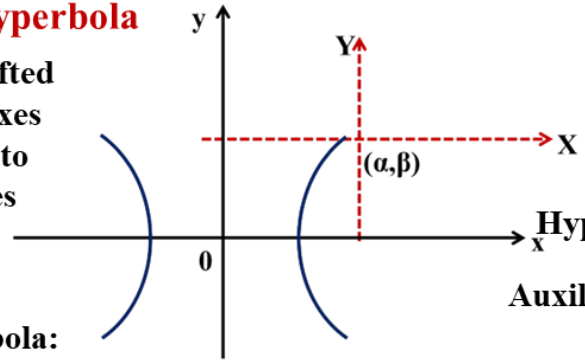
8. Shifted Hyperbola

If center is shifted to (α, β) and axes being parallel to coordinate axes



Eq. of Hyperbola:

$$\frac{(X)^2}{a^2} - \frac{(Y)^2}{b^2} = 1 \qquad \boxed{\frac{(x-\alpha)^2}{a^2} - \frac{(y-\beta)^2}{b^2} = 1}$$

9. Generalized Version

Eq. of transverse axis $\ell x + my + n = 0$

Eq. of conjugate axis $mx - \ell y + p = 0$

Length of transverse axis $= 2a$

Length of conjugate axis $= 2b$

Equation of the Hyperbola : $\frac{(X)^2}{a^2} - \frac{(Y)^2}{b^2} = 1$

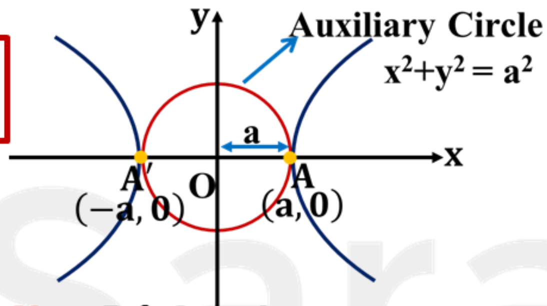
$$\boxed{\Rightarrow \frac{\left(\frac{mx - \ell y + p}{\sqrt{\ell^2 + m^2}}\right)^2}{a^2} - \frac{\left(\frac{\ell x + my + n}{\sqrt{\ell^2 + m^2}}\right)^2}{b^2} = 1}$$

10. Auxiliary Circle

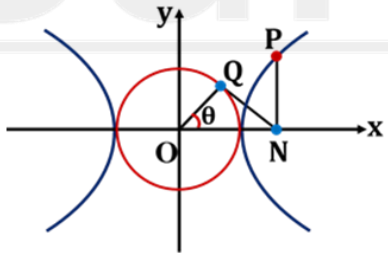
Circle described on the transverse axis as diameter is called the Auxiliary Circle of the hyperbola

Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Auxiliary Circle : $\boxed{x^2 + y^2 = a^2}$



Note: P & Q are known as corresponding points of each other.



$\angle QON$ = eccentric angle of point 'P'

$\angle QON = \theta \neq \angle PON$

$0 \le \theta < 2\pi, \theta \neq \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

11. Parametric Coordinates of Hyperbola

Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

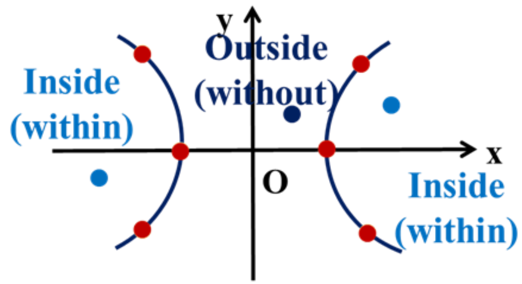
Parametric Coordinates :

$\boxed{P \equiv (a\sec\theta, b\tan\theta)}$ where θ is a parameter

θ	x	y	Point P lies on
$\left(0, \frac{\pi}{2}\right)$	+	+	Right Upper Branch
$\left(\frac{\pi}{2}, \pi\right)$	–	–	Left Lower Branch
$\left(\pi, \frac{3\pi}{2}\right)$	–	+	Left Upper Branch
$\left(\frac{3\pi}{2}, 2\pi\right)$	+	–	Right Lower Branch

12. Position of a Point w.r.t. Hyperbola

$S: \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$



- ❖ Point P lies on the Hyperbola. $S_1 = 0$
- ❖ Point P lies inside the Hyperbola. $S_1 > 0$
- ❖ Point P lies outside the Hyperbola. $S_1 < 0$

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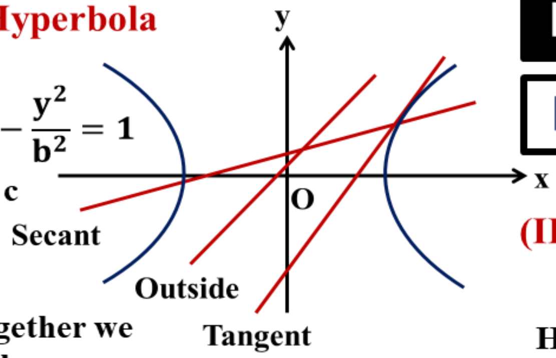
13. Line and Hyperbola

Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Line : $y = mx + c$



Solving them together we get a quad. eq where,

$D = 4a^2b^2[b^2 + c^2 - a^2m^2]$

$D > 0$	$D = 0$	$D < 0$
Secant	Tangent	Neither intersect nor touch

14. Tangent to the Hyperbola

(I) Cartesian Form

Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Point : (x_1, y_1)

Eq. of Tangent :

$$\boxed{\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1}$$

(II) Parametric form

Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Point : $(a.\sec\theta, b.\tan\theta)$

Eq. of Tangent in Parametric form

$$\boxed{\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1}$$

(III) Slope Form

Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Line : $y = mx + c$

Condition of tangency : $c^2 = a^2m^2 - b^2 (m \in R)$

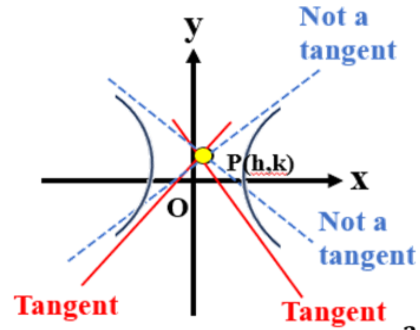
Eq. of Tangent : $\boxed{y = mx \pm \sqrt{a^2m^2 - b^2}}$

Tangent from an External Point

If tangent passes through (h, k) then

$k = mh \pm \sqrt{a^2m^2 - b^2}$

On simplifying we get a quad. Eq. in 'm'. Hence, passing through an external point there can be max. 2 tangents.

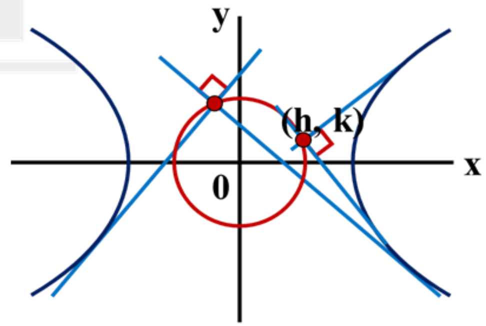


$m_1 + m_2 = \frac{2kh}{h^2 - a^2}$ $m_1 m_2 = \frac{k^2 + b^2}{h^2 - a^2}$

15. Director Circle

The locus of point of intersection of pair of $\perp$ tangents is called director circle.

$$\boxed{x^2 + y^2 = a^2 - b^2}$$



Note :

$x^2 + y^2 = a^2 - b^2 > 0, < 0, = 0$

(not possible)

If $\ell(TA) > \ell(CA)$ i.e. $a > b$

Director Circle is real with finite radius

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Hyperbola

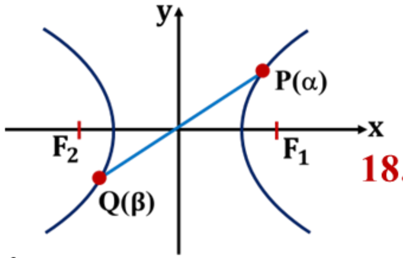
16. Chord of Hyperbola - Parametric

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Points:

$P(\alpha) \equiv (a\sec\alpha,\ b\tan\alpha)$

$Q(\beta) \equiv (a\sec\beta,\ b\tan\beta)$



Equation of the chord PQ :

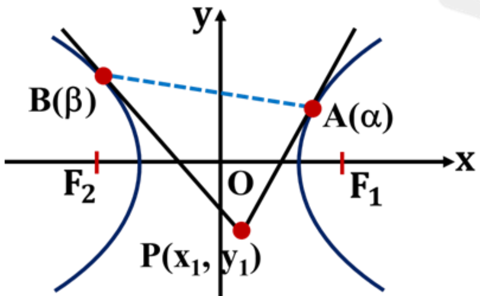
$$\boxed{\frac{x}{a}\cos\frac{(\alpha-\beta)}{2} - \frac{y}{b}\sin\frac{(\alpha+\beta)}{2} = \cos\frac{(\alpha+\beta)}{2}}$$

Note :

If this particular chord passes through (d, 0)

$$\boxed{\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{a-d}{a+d}}$$

17. Point of Intersection of Tangents



Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

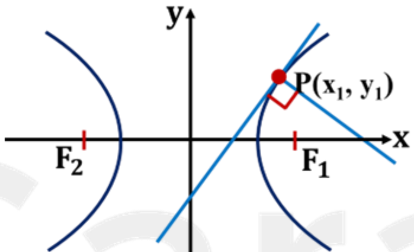
Point of intersection of tangents at $A(\alpha)$ & $B(\beta)$:

$$\boxed{x_1 = a\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}},\ y_1 = b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}}}$$

18. Normal to Hyperbola -

Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(i) Cartesian Form P: (x_1, y_1)



Equation of Normal at point P :

$$\boxed{\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2}$$

(ii) Parametric Form

Point : $P(a\sec\theta, b\tan\theta)$

$$\boxed{\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2}$$

(iii) Slope form :

$$\boxed{y = mx \pm \left(\frac{(a^2+b^2)m}{\sqrt{a^2 - b^2m^2}}\right)}$$

Parametric Form : $\boxed{\frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a}$

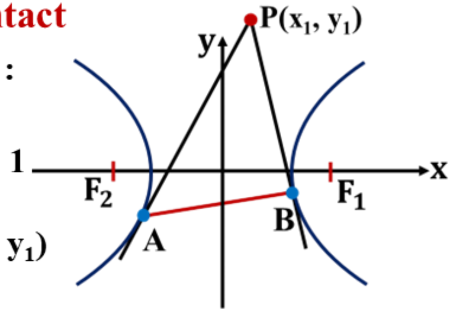
19. Pair of Tangents & Chord of Contact

Hyperbola :

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Point : (x_1, y_1)

Outside



$\boxed{\text{Pair of tangents: } SS_1 = T^2}$ Where,

$S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ $\quad S_1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

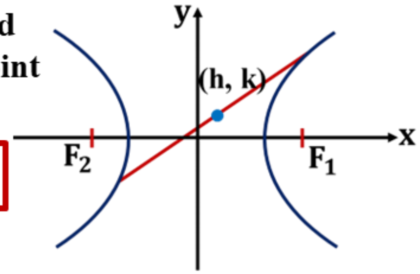
$T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

$\boxed{\text{Chord of contact: } T = 0}$ $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

20. Chord with given Mid-Point

Eq. of Chord with mid point (h, k)

$$\boxed{T = S_1}$$



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21. Rectangular Hyperbola

Equation of Rectangular Hyperbola:

$\dfrac{x^2}{a^2} - \dfrac{y^2}{a^2} = 1$ $\boxed{\Rightarrow x^2 - y^2 = a^2}$

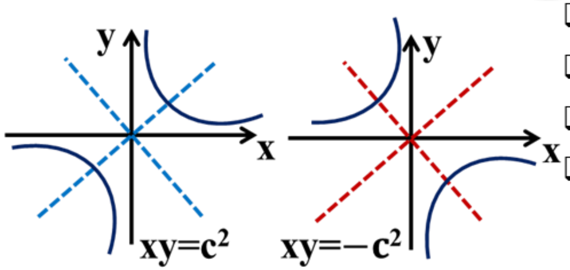
Note: In rectangular hyperbola asymptotes are $\perp$ to each other & their equations are

$\boxed{y = \pm x}$

❖ Equilateral Hyperbola $\Leftrightarrow$ Rectangular Hyperbola

❖ If a Hyperbola is equilateral then its Conjugate Hyperbola is also equilateral.

22. Rectangular Hyperbola ($xy = c^2$)



If the axes of rectangular Hyperbola are rotated by an angle $-\dfrac{\pi}{4}$ or $\dfrac{\pi}{4}$ about the same origin then,

Asymptotes are coordinates axes: $xy = 0$

Equation of the Rectangular Hyperbola is reduced to $xy = k = \dfrac{a^2}{2}$ or $xy = -\dfrac{a^2}{2}$

i.e. $xy = c^2$ or $xy = -c^2$

- ❑ Eccentricity $= \sqrt{2}$
- ❑ Asymptotes: $x = 0;\ y = 0$
- ❑ Transverse axis : $y = x$
- ❑ Conjugate Axis : $y = -x$ ❑ Centre : $(0, 0)$
- ❑ Vertex : (c, c) and $(-c, -c)$
- ❑ Foci : $(c\sqrt{2}, c\sqrt{2})$ & $(-c\sqrt{2}, -c\sqrt{2})$
- ❑ Length of Latus Rectum : $2\sqrt{2}c$
- ❑ Eq. of Auxiliary Circle : $x^2 + y^2 = 2c^2$
- ❑ Eq. of Director Circle : $x^2 + y^2 = a^2 - b^2 = 0$
- ❑ Equation of Directrices

Eq. of D_1: $\left(y - \dfrac{c}{\sqrt{2}}\right) = -1\left(x - \dfrac{c}{\sqrt{2}}\right)$

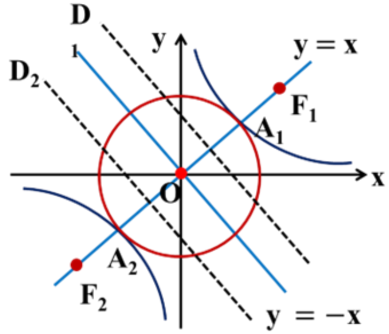
$\Rightarrow x + y = \sqrt{2}c$

Eq. of D_2: $x + y = -\sqrt{2}c$

- ❑ Parametric Coordinates

$x = ct,\ y = \dfrac{c}{t}$

Where, $t \in R - \{0\}$



23. Analysis of Rectangular Hyperbola

Rectangular Hyperbola : $xy = c^2$

Parametric Coordinates : $x = ct,\ y = \dfrac{c}{t}$

(i) Chord Joining Two Points $P(t_1)$ & $Q(t_2)$

$x + t_1t_2 . y = c(t_1 + t_2)$

Where, Slope $m = -1/(t_1t_2)$

(ii) Tangents -

Cartesian Form: $\dfrac{x}{x_1} + \dfrac{y}{y_1} = 2$

Parametric Form: $\dfrac{x}{t} + ty = 2c$ Slope $= -\dfrac{1}{t^2}$

(iii) Normal - Parametric Form:

Eq. of the normal : $xt^3 - yt = c(t^4 - 1)$

Note : If $P(t_1), Q(t_2), R(t_3)$ & $S(t_4)$ are four co-normal points then $t_1.t_2.t_3.t_4 = -1$.

(iv) Chord with given Mid Point

Mid Point : (h, k)

Eq. of Chord: $T = S_1 \Rightarrow kx + hy = 2hk$