MIND MAP

Matrices

1. Matrix

Arrangement of $m \times n$ elements into m – rows and n – columns is called matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{Vmatrix}$$

2. Types of Matrices

❑ **Row Matrix (Row Vector)**

A matrix having only one row is called a row matrix. Order : $(1 \times n)$

❑ **Column Matrix (Column Vector)**

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{bmatrix}$$

A matrix having only one column is called a column matrix.

Order : $(n \times 1)$

❑ **Zero or Null Matrix ($A = O_{m\times n}$) (Denoted by 'O')**

A $m \times n$ matrix whose all entries are zero is known as a zero or null matrix.

❑ **Horizontal Matrix**

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}_{2\times 4}$$

A matrix of order $m \times n$ is a horizontal matrix

if $n > m$

❑ **Vertical Matrix**

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 2 & 5 \\ 1 & 1 \end{bmatrix}_{4\times 2}$$

A matrix of order $m \times n$ is a vertical matrix

if $m > n$

❑ **Singleton Matrix**

A matrix having only one element is known as singleton matrix.

❑ **Square Matrix (Order n)**

If number of rows = number of columns

The matrix is known as square matrix.

Note

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

1. The elements $a_{11}, a_{22}, a_{33}, \ldots\ldots$ a_{nn} are called Diagonal Elements.

The line along which the diagonal elements lie is called "Principal or Leading" diagonal.

2. In a square matrix the pair of elements a_{ij} & a_{ji} are called Conjugate Elements.

$a_{12} \longleftrightarrow a_{21}$ $\quad a_{13} \longleftrightarrow a_{31}$ $\quad a_{23} \longleftrightarrow a_{32}$

3. Determinant of matrix exist only when it is square matrix.

❑ **Diagonal Matrix**

$$A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \quad a_{ij} = 0 \quad \text{if } i \neq j$$

In a square matrix if all the elements except diagonal elements are zero it is called Diagonal Matrix.

❑ **Scalar Matrix**

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \text{If } d_1 = d_2 = d_3 = a$$

In a diagonal matrix, if all the elements of principal diagonal are equal it is known as Scalar Matrix.

❑ **Unit or Identity Matrix**

(Denoted by 'I')

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{If } a = 1$$

$$\text{Det (I)} = |I| = 1$$

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❑ Triangular Matrix

(a) Upper Triangular Matrix

$$\begin{bmatrix} X & X & X \\ 0 & X & X \\ 0 & 0 & X \end{bmatrix} \quad \text{If } a_{ij} = 0 \ \forall \ i > j$$

(b) Lower Triangular Matrix

$$\begin{bmatrix} X & 0 & 0 \\ X & X & 0 \\ X & X & X \end{bmatrix} \quad \text{If } a_{ij} = 0 \ \forall \ j > i$$

3. Trace of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$tr(A) = \sum a_{ii} = a_{11} + a_{22} + a_{33}$$

Sum of principal diagonal elements of a square matrix is known as Trace of a Matrix.

Properties of tr(A)

1. $tr.\ (\lambda A) = \lambda\ tr\ (A)$
2. $tr\ (A + B) = tr\ (A) + tr(B)$
3. $tr(AB) = tr(BA)$

4. Algebra of Matrices

Equality of Matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$$

The matrices A & B are equal if :

$a = k$ $\quad c = m$
$b = l$ $\quad d = n$ $\quad A = [a_{ij}] \quad B = [b_{ij}]$

(i) both have the same order.

(ii) $a_{ij} = b_{ij}$ for each pair of i & j.

Addition of Matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} k & l \\ m & n \end{bmatrix}$$

$$A = [a_{ij}]_{m \times n} \quad B = [b_{ij}]_{m \times n}$$

Then,

$$A + B = \begin{bmatrix} a+k & b+l \\ c+m & d+n \end{bmatrix}$$

$$A + B = [a_{ij} + b_{ij}]$$

Where, A & B are of the same order.

Properties of Addition

Additive Inverse

If $A + B = O = B + A$

& both A and B have the same order

Then, A and B are said to be the additive inverse of each other.

Where, O is the null matrix of the same order as that of A and B.

$A + (-A) = O$

Existence of Additive Identity

$A + O = O + A = A$

The zero matrix plays the same role in matrix addition as the number zero does in addition of numbers.

Cancellation Law

If, $A + B = A + C \Rightarrow B = C$

(Left cancellation)

If, $B + A = C + A \Rightarrow B = C$

(Right cancellation)

$\therefore$ Cancellation laws hold good.

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Matrices

Note

$A = [a_{ij}]_{m\times n}$ $\quad B = [b_{ij}]_{x\times p}$

1. In the product of matrices, AB exists, but BA may or may not exist.

2. $(AB)_{ij} = \sum_{r=1}^{n} a_{ir}.b_{rj}$

A matrix of order $m \times p$.

Properties of Multiplication

Positive Integral Powers of a Square Matrix

For a square matrix A, $A.A = A^2$

$A.A.A = (A.A).A = A^2.A = A^3$

(i) $A^m A^n = A^{m+n}$

(ii) $(A^m)^n = A^{mn}$ (for $m,n \in N$)

Note ❖ $(A+B)^2 = (A+B)(A+B)$

$= A^2 + AB + BA + B^2$

If A & B commute	If A & B anticommute
(ie. $AB = BA$)	(ie. $AB = -BA$)
$A^2 + 2AB + B^2$	$A^2 + B^2$

In general $(A+B)^2 \neq A^2 + B^2 + 2AB$

❖ In general

$(A+B)^n = {}^nC_0.A^n + {}^nC_1.A^{n-1}.B + ... + {}^nC_n.B^n$

$(n \in N)$ (if A & B commute each other)

❖ $A^2 - B^2 = (A+B)(A-B)$

if A and B commute.

❖ If $A = \text{dia}\ [d_1\ d_2\ d_3]$

Then, $A^n = \text{dia}\ [d_1^n\ d_2^n\ d_3^n]$

Corresponding determinants of square matrices

For square matrices A & B,

i) $|AB| = |A|\ |B|$

In general $|A^k| = |A|^k$, $k \in N$

ii) $|KA| = K^n|A|$ Where, n is order of A

5. Multiplication of Matrices

If, $A = [a_{ij}]_{m\times n}$ $\quad B = [b_{ij}]_{x\times p}$

Then AB exists only if,

No. of columns in pre multiplier (A) $\quad x = n$

= No. of rows in post multiplier (B)

6. Multiplication of Matrix by a Scalar

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ Then, $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

Note

If A is a square matrix then,

1. $tr(kA) = k[tr(A)]$ 2. $|kA| = k^n|A|$

Where, n is order of matrix A.

Properties of Matrix Multiplication

1. Matrix Multiplication is not Commutative (In General)

$\Rightarrow AB \neq BA$ (in general)

Note

❖ If A and B are two non – zero matrices such that $AB = 0$ then,

(a) $\not\Rightarrow A = 0$ or $B = 0$

(b) also $AB = 0 \not\Rightarrow BA = 0$

A and B are called the divisors of zero

❖ If $AB = AC \not\Rightarrow B = C$

However, If $B = C$ then, $AB = AC$

❖ If A is a square matrix then,

$I.A = A.I = A$

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Properties of Matrix Multiplication

2. Matrix Multiplication is Associative

If A, B & C are conformable for the product AB & BC,

Then, $(AB)C = A(BC)$

3. Distributivity

Matrix multiplication is distributive over addition.

Then, $A(B + C) = AB + AC$

$(A + B)C = AC + BC$

7. Matrix Polynomial

Introduction

If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_n$

$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + ... + a_n I_n$

Where, A = Square matrix

If $f(A)$ = Null matrix, then A is called zero or root of the polynomial $f(A)$.

Characteristic Equation

(Hamilton's Rule)

Let A be a square matrix.

Then,

$|A - xI| \rightarrow$ (characteristic poly of A)

$|A - xI| = 0 \rightarrow$ (characteristic eq. of A)

Note : For a square matrix A.

- tr(A) = sum of the roots of the characteristic equation
- det(A) = product of the roots of the characteristic equation

Hence, Characteristic Equation :

$A^2 - [tr(A)]A + det(A)I = 0$

8. Transpose of A Matrix

Transpose of a Matrix is obtained by changing its rows into columns and columns into rows.

$A = [a_{ij}]_{m \times n}$

Then, $A^T = [a_{ji}] = A'_{n \times m}$

Properties of Transpose

Matrices : A and B Transpose : A^T & B^T

1. $(A^T)^T = A$

Matrices : A and B

Transpose : A^T & B^T

2. $(A + B)^T = A^T + B^T$

(A & B have the same order.)

3. $(KA)^T = KA^T$

K be any scalar (real or complex)

4. $(A B)^T = B^T A^T$

Provided, A & B are conformable for matrix product AB

Hence, generally :

$(A_1.A_2. .... A_n)^T = A_n^T . ... A_2^T .A_1^T$

(reversal law for transpose)

5. $(A^n)^T = (A^T)^n$

6. $tr(A^T) = tr(A)$

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Orthogonal Matrices

A square matrix is said to be orthogonal matrix, if

7. $|A^T| = |A|$

$AA^T = I = A^T A$

9. Symmetric & Skew Symmetric Matrix

Symmetric Matrix

In matrix $A = [a_{ij}]$ if, $a_{ij} = a_{ji} \quad \forall\, i \,\&\, j$

Then, the matrix is said to be symmetric matrix. & $A = A^T \Rightarrow A - A^T = 0$

Skew Symmetric Matrix

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if,

$a_{ij} = -a_{ji} \quad \forall\, i \,\&\, j \;\&\; a_{ii} = 0$

e.g.

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{bmatrix} \begin{matrix} \Rightarrow A = -A^T \\ \Rightarrow A + A^T = 0 \end{matrix}$$

Note

For a skew symmetric matrix A of order $(2p - 1)$ i.e. odd, $|A| = 0$.

Properties of Symmetric & Skew Symmetric Matrix

1. If A be a square matrix then,

a) $A + A^T$ is a symmetric matrix

b) $A - A^T$ is a skew symmetric matrix

c) $A A^T$ and $A^T A$ are symmetric matrices.

2. Every square matrix can be uniquely expressed as the sum of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{Skew Symmetric}}$$

10. Adjoint of a Square Matrix

If, $A = [a_{ij}]$ And, $C = [C_{ij}]$

Then,

adj A = Transpose of the matrix of cofactors of elements of A in det(A)

Note

1. If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

Then, adj $A = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$

2. The adjoint of a scalar matrix is also a scalar matrix.

3. Adjoint of a diagonal matrix
Adjoint of a triangular matrix
} Triangular matrix.

Properties of Adjoint

$A = [a_{ij}]$

Square Matrix

1. $A\,(\text{adj. } A) = (\text{adj. } A).A = |A|\, I_n$.

$A = [a_{ij}]_n$

Non-Singular Matrix

2. $|\text{adj } A| = |A|^{n-1}$

$A = [a_{ij}]_n$

3. $\text{adj}(KA) = K^{n-1}\,\text{adj}(A)$

$A = [a_{ij}]_n \quad B = [b_{ij}]_n$

non singular matrices

4. adj AB = (adj B) (adj A)

$A = [a_{ij}]_n$

Invertible Matrix

5. $\text{adj } A^T = (\text{adj } A)^T$

$A = [a_{ij}]_n$

Non – Singular Matrix

6. (a) $\text{adj}\,(\text{adj } A) = |A|^{n-2} A$

(b) $|\text{adj}\,(\text{adj } A)| = |A|^{(n-1)^2}$

(c) $|\text{adj}(\text{adj}(\text{adj}A)| = |A|^{(n-1)^3}$

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11. Inverse of a Matrix

A square matrix 'A' is said to be invertible (non singular) if there exist a matrix B such that :

$AB = I = BA$

$B = A^{-1}$ (Inverse of A)

11. Inverse of a Matrix

Thus, $A^{-1} = B \Leftrightarrow AB = I = BA$

$$A^{-1} = \frac{(\text{adj}A)}{|A|}$$

Note

1. For a Square matrix A to be invertible.
   $|A| \neq 0$

2. Let, $A = \text{dia}\,(k_1\, k_2\, k_3 \ldots\ldots k_n)$
   (Non Singular)
   Then,
   $A^{-1} = \text{dia}\left(k_1^{-1},\ k_2^{-1},\ k_3^{-1}\ldots\ldots, k_n^{-1}\right)$

Some Important Theorems

1. Every invertible matrix posses a unique inverse.

2. If, $A \rightarrow$ Invertible & Square
   Then,
   $A^T \rightarrow$ Invertible & $(A^T)^{-1} = (A^{-1})^T$

3. If, $A \rightarrow$ Non – Singular
   Then, $|A^{-1}| = |A|{-1}$ i.e. $|A^{-1}| = \dfrac{1}{|A|}$

4. If A & B are Invertible Matrices
   Then, $(AB)^{-1} = B^{-1}A^{-1}$

5. If, $A \rightarrow$ Invertible
   Then, (a) $(A^{-1})^{-1} = A$
   (b) $(A^k)^{-1} = (A^{-1})^k = A^{-k}$

12. Special Matrices

❑ Idempotent Matrix

A square matrix is said to be idempotent matrix if, $A^2 = A$

For an idempotent matrix A,
$A^n = A\ \forall\ n \geq 2,\ n \in N$

Note

1. For Idempotent matrices $|A|^2 = |A|$
   $|A| = 0$ OR $|A| = 1$

2. The only invertible idempotent matrix is I.

❑ Nilpotent Matrix

A square matrix is said to be nilpotent matrix of index p, $(p \in N)$,

If $A^p = O, A^{p-1} \neq O$

Where,

p is the least positive integer for which $A^p = O$.

Note

The value of determinant of nilpotent matrix is 0.

❑ Periodic Matrix

A square matrix which satisfies the relation $A^{K+1} = A$, for some positive integer K is known to be a periodic matrix with period K.

Period of an idempotent matrix is 1.

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❑ Involuntary Matrix

If $A^2 = I$, the matrix is said to be an involuntary matrix.

In other words square root of unit matrix is called involuntary matrix.

13. App. of Matrices & Determinants

Sol. of System of Linear Equations

$a_1x + b_1y + c_1z = d_1$ .... (i)

$a_2x + b_2y + c_2z = d_2$ .... (ii)

$a_3x + b_3y + c_3z = d_3$ .... (iii)

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then,

$AX = B \Rightarrow X = A^{-1}B$

$$X = \frac{\text{adj}A}{|A|}.B$$

Types of solutions

- $|A| \neq 0$
  (Unique Solution)
  (Consistent)
- $|A| \neq 0$ & $(\text{adj}\,A).B \neq 0$
  (Unique non−trivial sol.)
  (Consistent)
- $|A| \neq 0$ & $(\text{adj}\,A).B = 0$
  (Trivial sol. $X = 0$)
  (Consistent)

$$X = \frac{\text{adj}A}{|A|}.B$$

Types of solutions

If $|A| = 0$

- $(\text{adj}\,A).B = 0$
  Infinite sols.
  Consistent
- $(\text{adj}\,A).B \neq 0$
  No solution
  Inconsistent