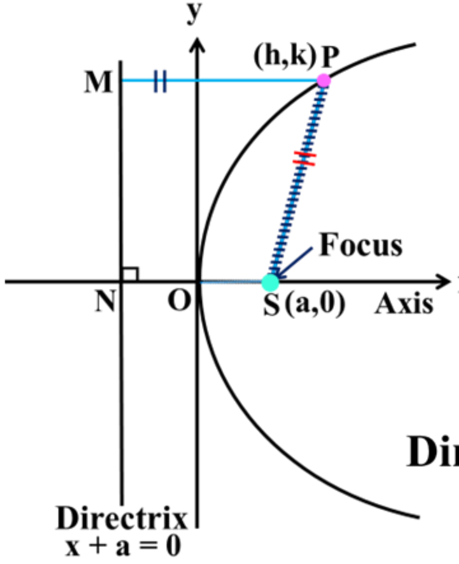
MIND MAP

Parabola

1. Equation of Standard Parabola

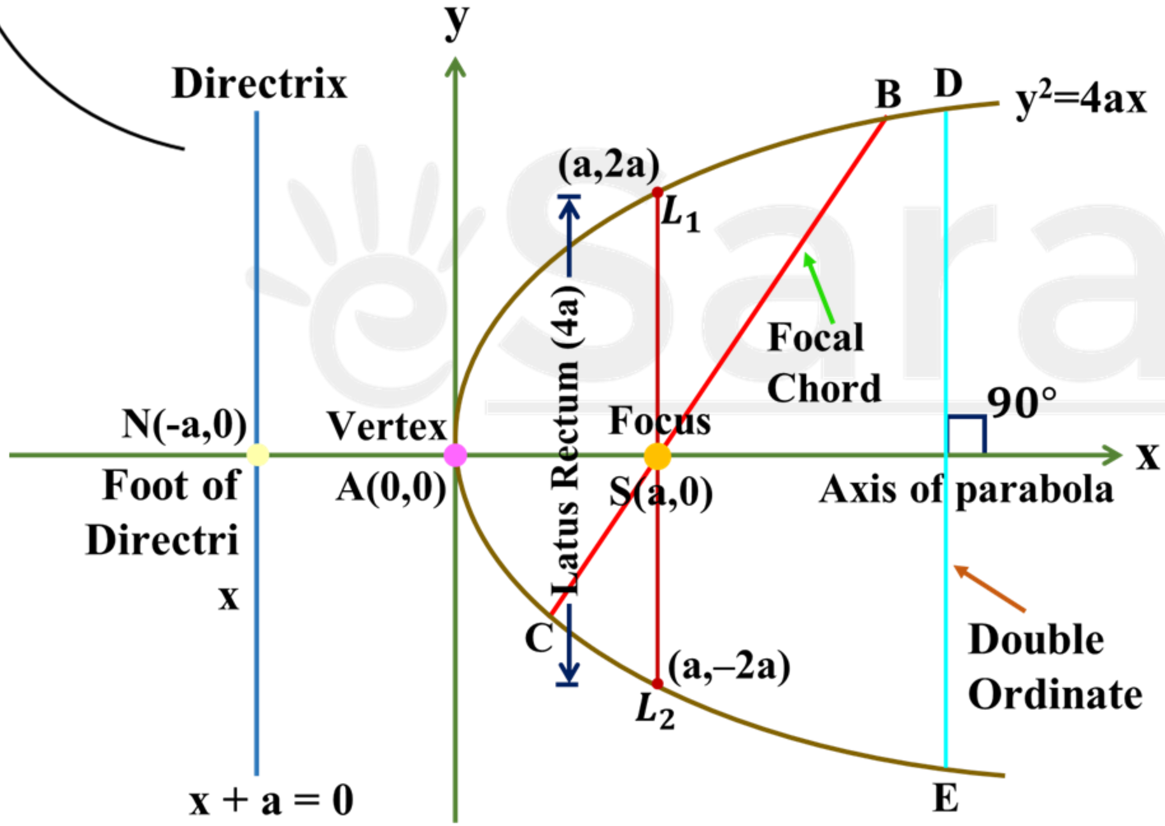


Using the property of parabola : PS = PM

Eq. of Standard Parabola :

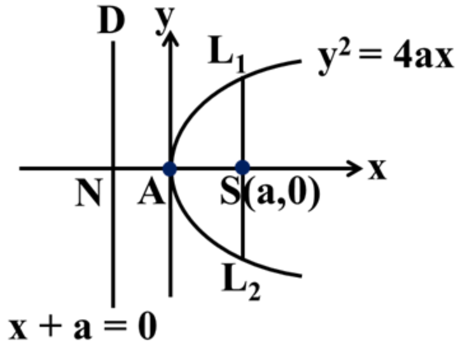
$y^2 = 4ax$

2. Parabola at a Glance

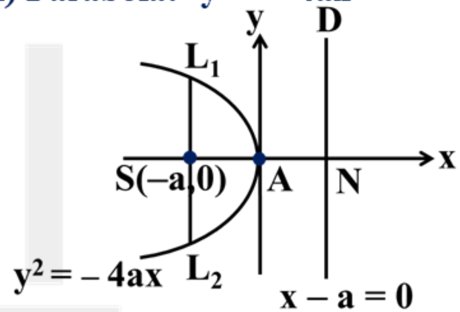


3. Four Standard Parabolas

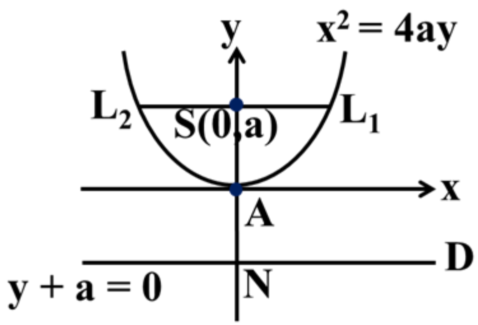
(i) Parabola: $y^2 = 4ax$



(ii) Parabola: $y^2 = -4ax$



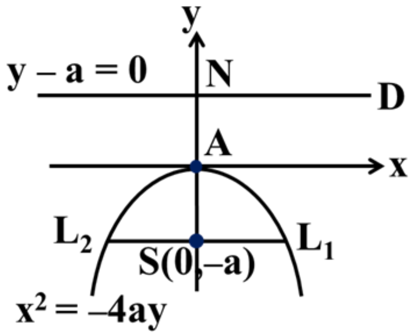
(iii) Parabola: $x^2 = 4ay$



MIND MAP
Parabola

3. Four Standard Parabolas

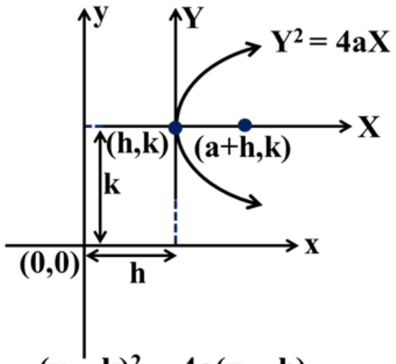
(iv) Parabola: $x^2 = -4ay$



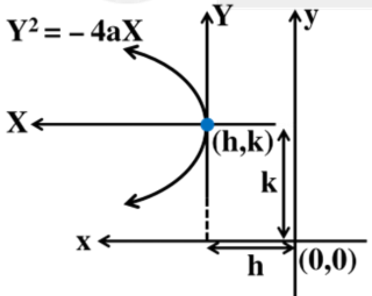
4. Shifted Parabola

Case (a)

When vertex of the parabola is not origin but shifted to (h, k) and its axis is parallel to x-axis.



$(y-k)^2 = 4a(x-h)$



$(y-k)^2 = -4a(x-h)$

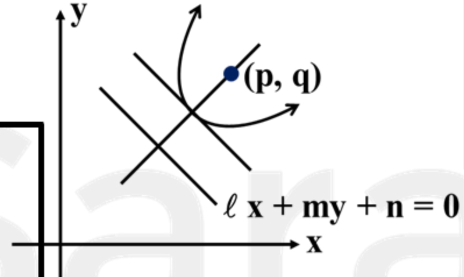
Case (b)

General Parabola : Focus (p, q) and directrix :

$lx + my + n = 0$

A Parabola is the locus of a point having equal distances from Focus and Directrix.

$$(x-p)^2 + (y-q)^2 = \frac{(\ell x + my + n)^2}{\ell^2 + m^2}$$



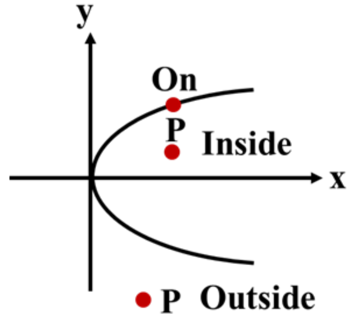
5. Position of a Point Relative To a Parabola

Let $P \equiv (x_1, y_1)$ $\quad S \equiv y^2 - 4ax = 0$ $\quad S_1 \equiv y_1^2 - 4ax_1$

If $S_1 < 0$ Point P lies inside the parabola.

If $S_1 = 0$ Point P lies on the parabola.

If $S_1 > 0$ Point P lies outside the parabola.



6. Parametric Coordinates

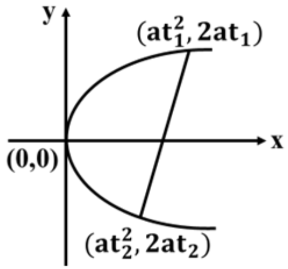
Parabola	Parametric Coordinates :
$y^2 = 4ax$	$x = at^2, y = 2at$
$y^2 = -4ax$	$x = -at^2, y = 2at$
$x^2 = 4ay$	$x = 2at, y = at^2$
$x^2 = -4ay$	$x = 2at, y = -at^2$
$(y-\beta)^2 = 4a(x-\alpha)$	$x = \alpha+at^2, y = \beta+2at$
$(x-\alpha)^2 = 4a(y-\beta)$	$x = \alpha+2at, y = \beta+at^2$

7. Chord Joining Two Points t_1 & t_2

Parametric equation of a chord joining two points 't_1' and 't_2' of the parabola $y^2 = 4ax$ is

$$\boxed{2x - y(t_1 + t_2) + 2at_1t_2 = 0}$$

For memory $x - y(\text{A.M.}) + a(\text{G.M.})^2 = 0$



MIND MAP

Parabola

Remarks:

(i) Length of the chord :

$$\boxed{= a|t_1 - t_2|\sqrt{(t_1+t_2)^2+4}}$$

(ii) If chord is Focal Chord (V. Imp.)

(OR t_1, t_2, $(a, 0)$ are collinear)

$\Rightarrow$ Chord passing through $(a, 0)$

$\Rightarrow \boxed{t_1t_2 = -1} \Rightarrow t_2 = -\frac{1}{t_1}$

Hence, if one end of the focal chord is 't', the other end is $-\frac{1}{t}$.

Coordinates of focal chord can be taken as $\boxed{(at^2, 2at) \text{ and } \left(\frac{a}{t^2}, -\frac{2a}{t}\right)}$

(Note : Only one variable)

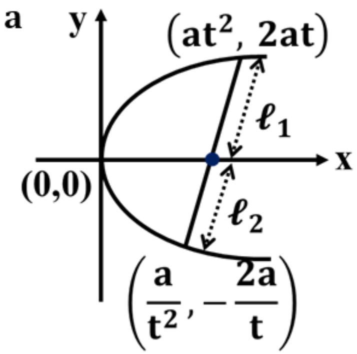
(iii) Length of the focal chord

$$\boxed{\ell_f = a(t_1 - t_2)^2 = a\left(t + \frac{1}{t}\right)^2}$$

(iv) If ℓ_1, ℓ_2 are the length segments of a focal chord made by focus then length of its latus rectum is $\frac{4\ell_1\ell_2}{\ell_1 + \ell_2}$ i.e. $= 4a$.

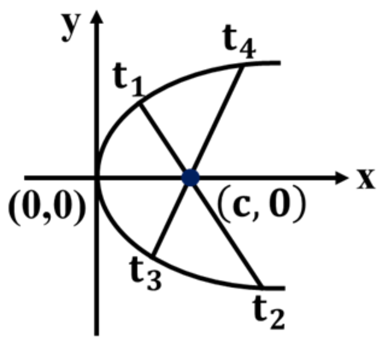
$\ell_1 = at^2 + a$ (use focal distance property)

$\ell_2 = \frac{a}{t^2} + a$



(v) If the chord joining 't_1', 't_2', and 't_3','t_4' passes through a point $(c, 0)$ on the axis, then

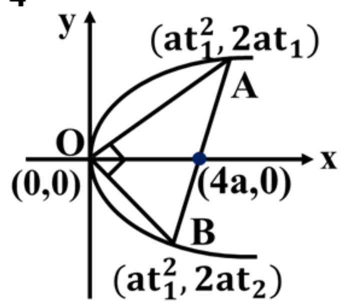
$$\boxed{t_1t_2 = t_3t_4 = -\frac{c}{a}}$$



(vi) If the chord subtends 90° angle at the vertex of parabola then $(t_1t_2 = -4)$ and it passes through a fixed point $(4a, 0)$ on the axis.

$m_1 = \frac{2}{t_1} \quad m_2 = \frac{2}{t_2} \quad m_1m_2 = -1$

$\Rightarrow t_1t_2 = -4$



$2x - y(t_1 + t_2) + 2at_1t_2 = 0$

(vii) If t_2 approaches to t_1 $(t_2 \to t_1)$

chords $\to$ tangent at t

$$\boxed{ty = x + at^2}$$

$\to$ equation of tangent at the point 't' of the parabola $y^2 = 4ax$.

MIND MAP

Parabola

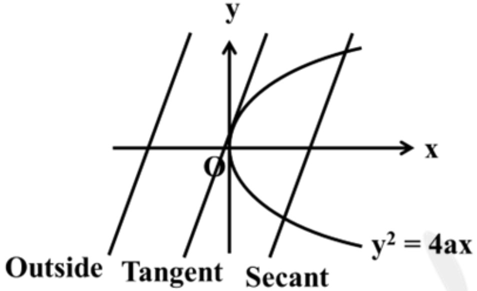
8. Line and Parabola

Parabola : $y^2 = 4ax$

Line : $y = mx + c$

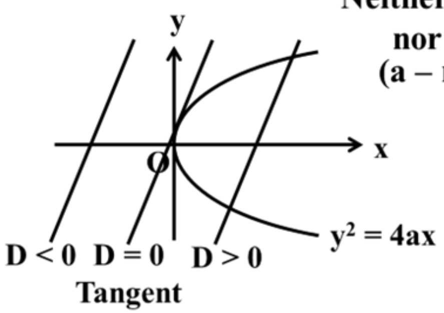
Solving them together $(mx + c)^2 - 4ax = 0$

$m^2x^2 + 2mcx + c^2 - 4ax = 0$



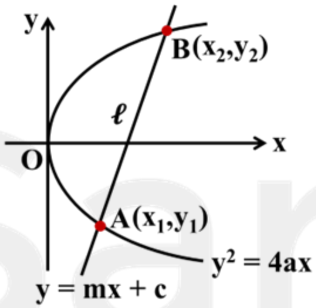
$D = 4(mc - 2a)^2 - 4m^2c^2$ $D = 16a(a - mc)$

- $D > 0$ Secant $(a - mc > 0)$
- $D = 0$ Tangent $(a - mc = 0)$
- $D < 0$ Neither intersect nor touch $(a - mc < 0)$



Length of Chord of Parabola Intercepted on Line

$$\boxed{\ell = \frac{4}{m^2}\sqrt{a(1 + m^2)(a - cm)}}$$



9. Tangents to The Parabola

(i) Point Form

Parabola :	$y^2 = 4ax$	$x^2 = 4ay$
Point :	(x_1, y_1)	(x_1, y_1)
Replace:	$y^2 \to yy_1$ & $x \to \frac{x + x_1}{2}$	$x^2 \to xx_1$ & $y \to \frac{y + y_1}{2}$
Equation of Tangent :	$yy_1 = 2a(x + x_1)$	$xx_1 = 2a(y + y_1)$

(ii) Parametric Form

Parabola : $y^2 = 4ax$	$x^2 = 4ay$
Point : $(at^2, 2at)$	Point : $(2at, at^2)$

Equation of tangent :

$y.2at = 2a(x + at^2)$	$2atx = 2a(y + at^2)$
$\Rightarrow \boxed{yt = x + at^2}$	$\Rightarrow \boxed{tx = y + at^2}$
Note Slope $m = 1/t$	Slope $m = t$

(iii) Slope Form

Tangent to The Parabola $x^2 = 4ay$

Parabola :	$y^2 = 4ax$	$x^2 = 4ay$
Line :	$y = mx + c$	$y = mx + c$
Condition of Tangency:	$c = \frac{a}{m}$	$c = -am^2$
Equation of Tangent :	$y = mx + \frac{a}{m}$	$y = mx - am^2$
Point of Contact:	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$(2am, am^2)$

(Tip for Point of contact : Interchange $x \leftrightarrow y$ and $m \leftrightarrow 1/m$)

MIND MAP

Parabola

10. Equation of tangents from any point outside the parabola

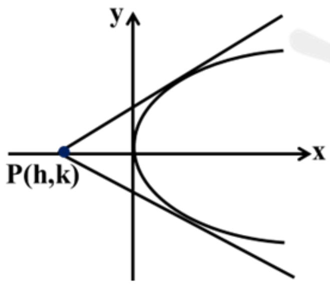
Equation of Tangent : $y = mx + \frac{a}{m}$

Point : (h, k)

$\Rightarrow k = mh + \frac{a}{m} \Rightarrow m^2h - mk + a = 0$

$\Rightarrow m_1 + m_2 = \frac{k}{h}$ & $m_1 m_2 = \frac{a}{h}$

These relations can be used to compute the locus of a point from which pairs of tangents are drawn.



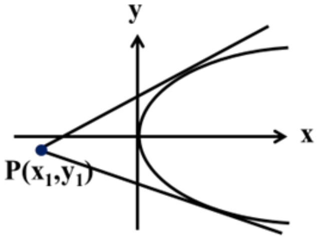
11. Equation of the pair of tangents

Parabola : $y^2 = 4ax$

Point : (x_1, y_1) (Outside)

Combined Equation of pair of tangents from the point (x_1, y_1)

$\boxed{SS_1 = T^2}$



Where,

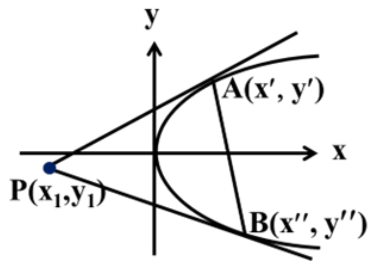
- $S \equiv y^2 - 4ax$
- $S_1 \equiv y_1^2 - 4ax_1$
- $T \equiv yy_1 - 2a(x + x_1)$

12. Chord of Contact

Equation of the chord of contact of pair of tangents from (x_1, y_1)

$T = 0$

$yy_1 = 2a(x + x_1)$



Note

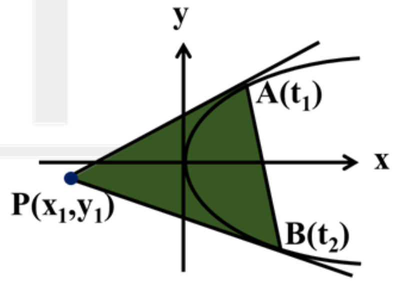
(i) Point of intersection of two tangents at t_1 & t_2

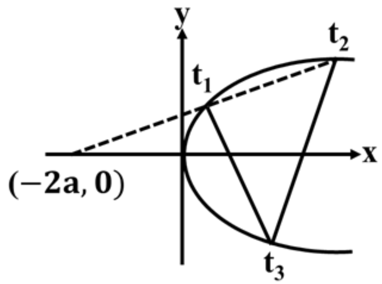
$(at_1t_2,\ a(t_1 + t_2))$

(G.M. of at_1^2 and at_2^2) (A.M. of $2at_1$ and $2at_2$)

(ii) Area of The Triangle PAB

$$\therefore \text{Area} = \frac{(y_1^2 - 4ax_1)^{3/2}}{2a} = \frac{(S_1)^{3/2}}{2a}$$





MIND MAP

Parabola

13. Director Circle

The locus of point of intersection of pair of perpendicular tangents is Director Circle.

OR

The locus of the points from which two perpendicular tangents can be drawn on the parabola.

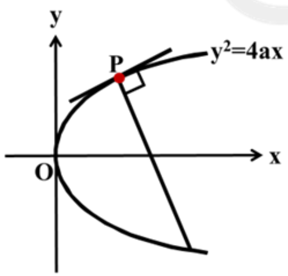
Director circle of the parabola is
$x + a = 0$
i.e. parabola's own directrix.

14. Normals To The Parabola

A line perpendicular to the tangent at a given point P is called Normal at point P

$$\text{Slope of Normal} = \frac{-1}{\text{Slope of tangent}}$$

$$\boxed{= \frac{-1}{\left(\frac{dy}{dx}\right)}}$$



(i) Cartesian Form/Point Form

Parabola : $y^2 = 4ax$

$\therefore$ Equation of Normal at the point (x_1, y_1)

$$\boxed{y - y_1 = -\frac{y_1}{2a}(x - x_1)}$$

(ii) Parametric Form

Parabola: $y^2 = 4ax$

Point : $(at^2, 2at)$

$\boxed{y = -tx + 2at + at^3}$ at $(at^2, 2at)$

With slope $m = -t$

(iii) Slope Form

Parabola : $y^2 = 4ax$ Point : $(at^2, 2at)$

Equation of normal at $(at^2, 2at)$

$y = -tx + 2at + at^3$ Slope $m = -t$

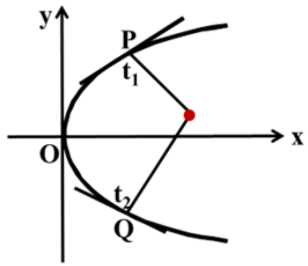
Replace t by $(-m)$

$\boxed{y = mx - 2am - am^3}$ At point $(am^2, -2am)$

Important Points

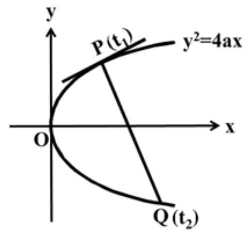
(i) Point of intersection of normals at t_1 & t_2

$(a(t_1^2 + t_2^2 + t_1t_2 + 2), \ -at_1t_2(t_1 + t_2))$



(ii) If the normals to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then

$$t_2 = -t_1 - \frac{2}{t_1}$$



(iii) If the normals to parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then

(i) $t_1t_2 = 2$ (ii) $t_3 = -(t_1 + t_2)$ and

(iii) The line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

15. Equation of Normal From a Point And Its Analysis

Parabola: $y^2 = 4ax$ Point: (h, k)

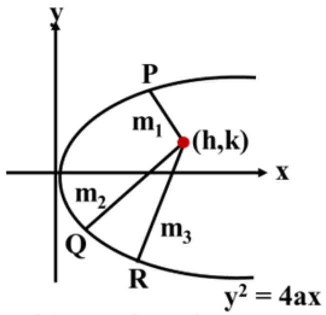
Method:

(i) Take the equation of normal in slope form $y = mx - 2am - am^3$

(ii) Put the given point (h, k)

(iii) Cubic equation in terms of m
$am^3 + (2a - h)m + k = 0$

(iv) Solve for m and find the equations of normals.



Analysis :

$am^3 + (2a - h)m + k = 0$

Let m_1, m_2 & m_3 are the slopes of the three concurrent normal, then

(i) $m_1 + m_2 + m_3 = 0$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - h}{a}$$

$$m_1m_2m_3 = -\frac{k}{a}$$

(ii) Foot of the normals of three concurrent normals are called conormals points.
At most three normals can be drawn from a given point.
Atleast one real normal can be drawn.

(iii) $P \equiv (am_1^2, -2am_1)\ Q \equiv (am_2^2, -2am_2)$
$R \equiv (am_3^2, -2am_3)$

(iv) $\because m_1 + m_2 + m_3 = 0$
Algebraic sum of the slopes of three concurrent normals is zero.
$-2am_1 - 2am_2 - 2am_3 = 0$

MIND MAP

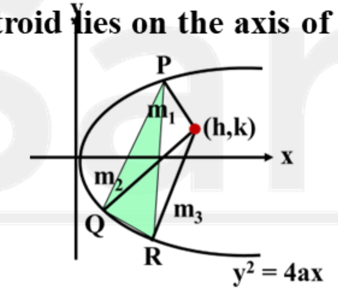
Parabola

Sum of ordinates of the three conormal points on the parabola is zero.

(v) $\because m_1 + m_2 + m_3 = 0$
Centroid of the triangle PQR

$$\equiv \left(\frac{\Sigma am_1^2}{3}, \frac{-2a\Sigma m_1}{3}\right) \equiv \left(\frac{\Sigma am_1^2}{3}, 0\right)$$

$\Rightarrow$ Centroid lies on the axis of parabola.



16. Equation of chord with a given Mid Point

Parabola: $y^2 = 4ax$ Mid Point: (h, k)

Equation of Chord with mid point (h, k)

$T = S_1 \Rightarrow \boxed{ky - 2a(x + h) = k^2 - 4ah}$

17. Diameter

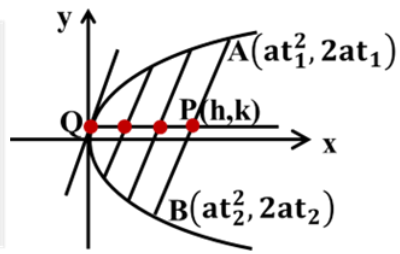
The locus of the middle points of a system of parallel chords of a Parabola is called its Diameter.

Eq. of the diameter is $y = \dfrac{2a}{m}$

Line parallel to the axis of the parabola

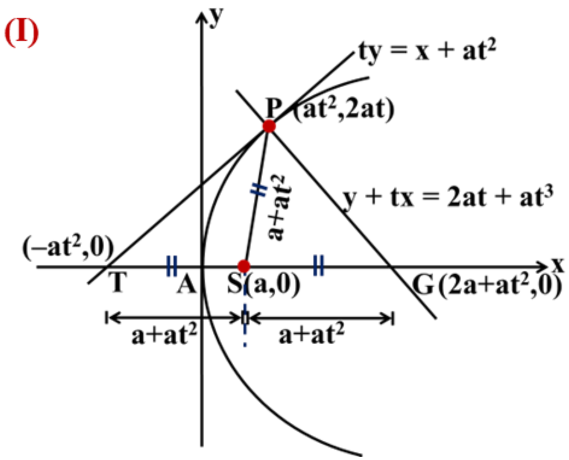
$$Q : \left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

And chord at Q becomes tangent to parabola.



Important Highlights of Parabola

(I)



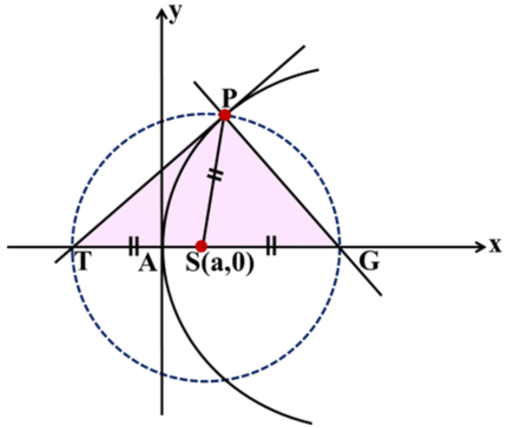
MIND MAP

Parabola

If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G, then

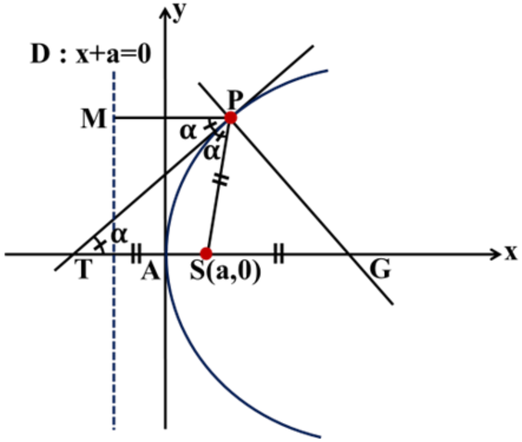
$ST = SG = SP$ where 'S' is the focus.

(II)



Circle circumscribing the triangle ΔTGP formed by any tangent, normal and x-axis, has its centre at focus.

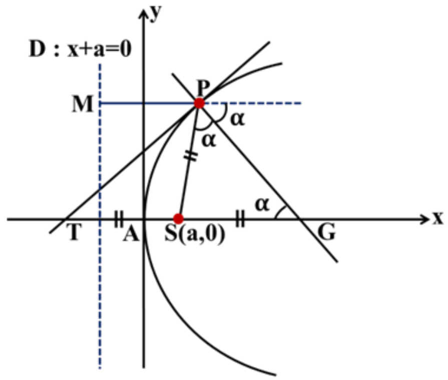
(iii)



The tangent at a point P on the parabola is the internal angle bisector of the angle between the focal radius SP & the perpendicular from P on the directrix.

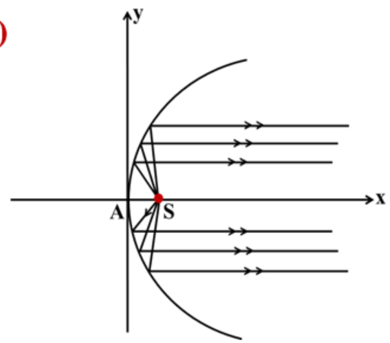
$\angle STP = \angle SPT = \angle TPM = \alpha$

(IV)



The normal at a point P on the parabola is the external angle bisector of the angle between the focal radius SP & the perpendicular from P on the directrix.

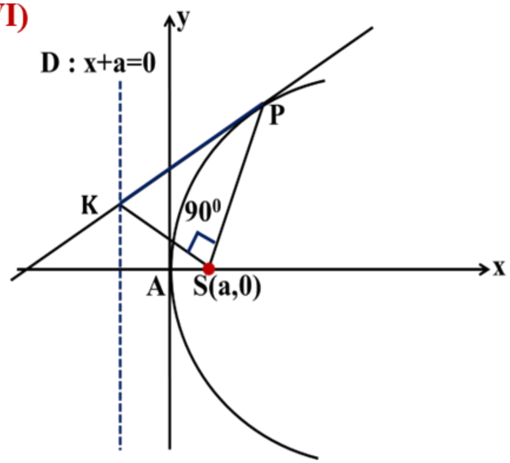
(V)



Reflection Property

All rays eminating from S will become parallel to the axis of the parabola after reflection.

(VI)



The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the focus.

$\Rightarrow \angle KSP = 90°$

(VII) $\Delta SPK \cong \Delta MPK$

$\because SP = PM \quad KP = KP$

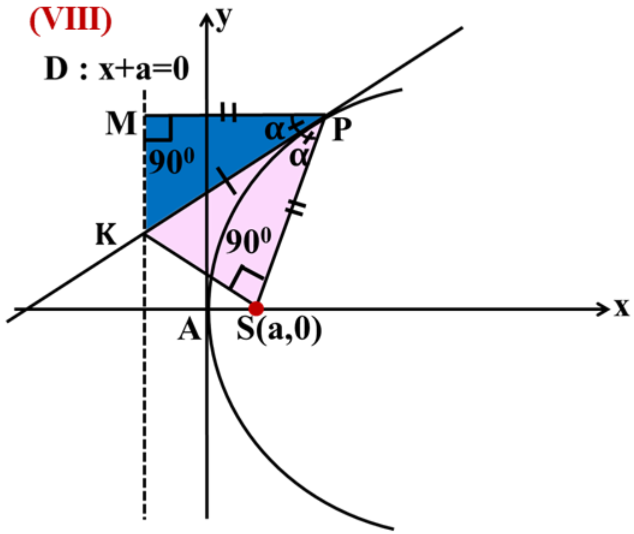
$\Rightarrow \angle PMK = \angle PSK = 90°$

$\angle\alpha = \angle\alpha$

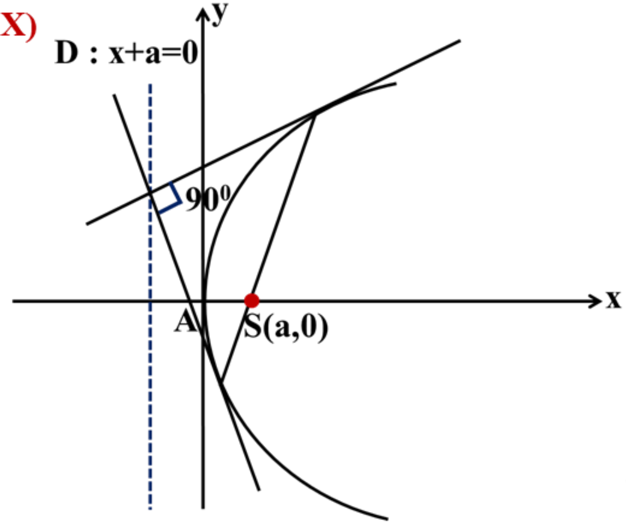
MIND MAP

Parabola

(VIII)

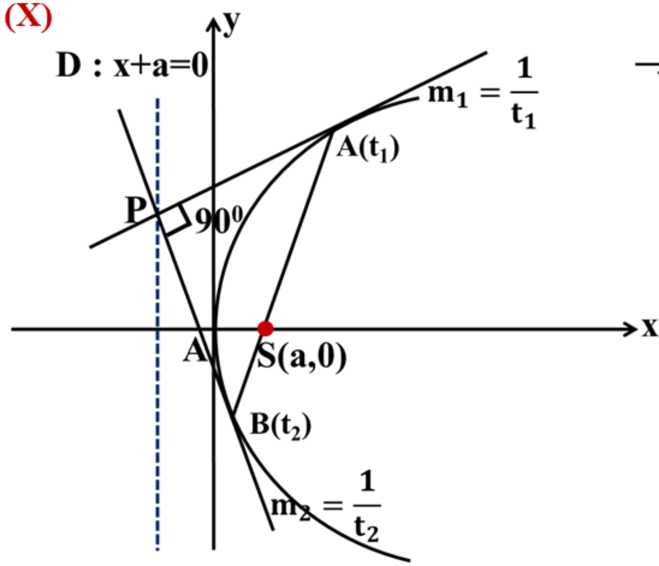


(IX)



The tangents at the extremities of a focal chord intersect at right angles on the directrix.

(X)



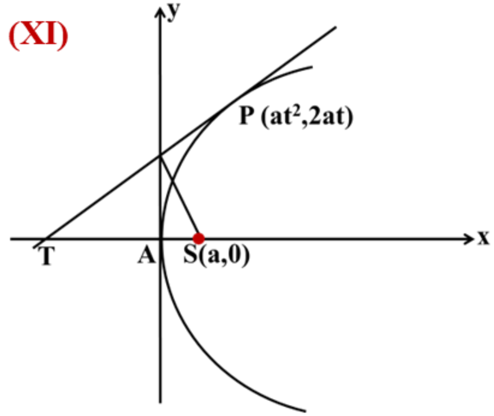
$$\Rightarrow m_1 m_2 = \frac{1}{t_1 t_2} = -1$$

Also x coordinate of point P :

$x = at_1t_2 = -a$

Chord of contact of tangents drawn from any point on directrix of parabola will be its focal chord and tangents will be at right angle.

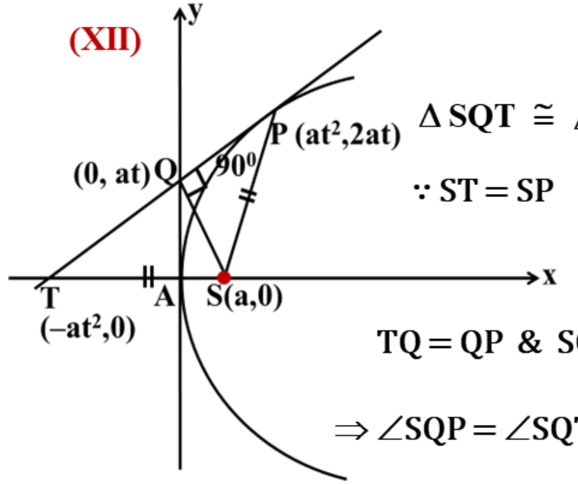
(XI)



Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex

Locus of the feet of the $\perp$ drawn from focus upon a variable tangent is the tangent drawn to the parabola at its vertex.

(XII)



$\Delta SQT \cong \Delta SQP$

$\because ST = SP$

$TQ = QP$ & $SQ = SQ$

$\Rightarrow \angle SQP = \angle SQT = 90°$