MIND MAP

Quadratic Equation

Polynomial

An expression of the type $P_n(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_n$ is called a polynomial of degree 'n', where all powers of x are non-negative integers and a_0 which is called leading coefficient of the polynomial should not be equal to zero.

$n = 1$, $P(x) = a_0x + a_1$, linear polynomial.
$n = 2$, $P(x) = a_0x^2 + a_1x + a_2$, quadratic polynomial.
$n = 3$, $P(x) = a_0x^3 + a_1x^2 + a_2x + a_3$, cubic polynomial.
$n = 4$, $P(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$, bi-quadratic poly.

$P_n(\alpha)$ means value of the polynomial $P_n(x)$ at $x = \alpha$
If $P_n(\alpha) = 0$, then α is called as zero of the polynomial.

Remainder Theorem

$\frac{P(x)}{x-k} = Q(x) + \frac{R}{x-k}$ where Q(x) is quotient & R is remainder.

$\Rightarrow P(x) = Q(x)(x-k) + R$ at $x = k$, $P(k) = R$

Factor Theorem

Let $P(x) = (x-k)\,Q(x) + R$

when $P(k) = 0$, $P(x) = (x-k)\,Q(x)$.

Therefore, P(x) is exactly divisible by $x - k$.

Quadratic Expression and Quadratic Equation

$\boxed{y = ax^2 + bx + c} \longrightarrow$ Quadratic Expression

where a = leading coefficient & c = absolute term of quadratic polynomial.

$\boxed{ax^2 + bx + c = 0;\ a \neq 0} \longrightarrow$ Quadratic Equation

Solving a quadratic equation means finding the values of x for which $ax^2 + bx + c$ vanishes and these values of x are also called the roots of quadratic equation.

Solution of quadratic equation

roots of $ax^2 + bx + c = 0$; $a \neq 0$; $a, b, c \in R$.

Hence $\alpha = \frac{-b+\sqrt{D}}{2a}$ and $\beta = \frac{-b-\sqrt{D}}{2a}$

where $D = b^2 - 4ac$

Relation between roots and coefficient

$ax^2 + bx + c = 0$; $a \neq 0$; $a, b, c \in R$

If α, β are the roots then $\alpha + \beta$

$= -\frac{b}{a}$; $\alpha\beta = \frac{c}{a}$ and $|\alpha - \beta| = \frac{\sqrt{D}}{|\alpha - \beta|}$

Formation of a quadratic equation when roots are given

Let α and β be the given roots of a quadratic equation, then

$(x-\alpha)(x-\beta) = 0 \Rightarrow x^2 - x(\alpha+\beta) + \alpha\beta = 0$

$x^2 - x$ (sum of the roots)+Product of the roots = 0

Nature of Roots

Consider the quadratic equation

$\boxed{ax^2 + bx + c = 0}$

where $a, b, c \in R$ and $a \neq 0$.

Roots of the equation are given by

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}}$$

(i) If $D > 0 \Rightarrow$ roots are real and distinct.
(ii) If $D = 0 \Rightarrow$ roots are equal.
(iii) If $D < 0 \Rightarrow$ roots are imaginary.

Important Note

(1) If co-efficients of the quadratic equation are rational then its irrational roots always occur in pair. If $p + \sqrt{q}$ is one of the roots then other root will be $p - \sqrt{q}$.

(2) If co-efficients of the quadratic equation are real then its imaginary roots always occur in complex conjugate pair. If $p + iq$ is one of the roots then other root will be $p - iq$.

Identity

Let $ax^2 + bx + c = 0$ be a equation. Now, if this equation has more than two distinct roots then it becomes an identity and in this case $a = b = c = 0$.

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Quadratic Equation

Symmetric Expressions

The symmetric expressions of the roots α, β of an equation are those expressions in α and β, which do not change by interchanging α and β.

(i) $\alpha^2+\beta^2$ (ii) $\alpha^2+\alpha\beta+\beta^2$ (iii) $\frac{1}{\alpha}+\frac{1}{\beta}$

(iv) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (v) $\alpha^2\beta+\beta^2\alpha$ (vi) $\left(\frac{\alpha}{\beta}\right)^2+\left(\frac{\beta}{\alpha}\right)^2$

(vii) $\alpha^3+\beta^3$ (viii) $\alpha^4+\beta^4$

Formation of Quadratic Equation Whose Roots are Symmetric Expression of α and β

If α, β are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are

(i) $-\alpha, -\beta \Rightarrow$ (Replace x by $-x$)

(ii) $\frac{1}{\alpha}, \frac{1}{\beta} \Rightarrow \left(\text{Replace x by } \frac{1}{x}\right)$

(iii) $\alpha^n, \beta^n, n \in N \Rightarrow \left(\text{Replace x by } x^{\frac{1}{n}}\right)$

(iv) $k\alpha, k\beta \Rightarrow \left(\text{Replace x by } \frac{x}{k}\right)$

(v) $k+\alpha, k+\beta \Rightarrow$ (Replace x by $(x-k)$)

(vi) $\frac{\alpha}{k}, \frac{\beta}{k} \Rightarrow$ (Replace x by kx)

(vii) $\alpha^{\frac{1}{n}}, \beta^{\frac{1}{n}}; n \in N \Rightarrow$ (Replace x by x^n)

Condition of common roots

Let $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have a common root α.

Hence $a_1\alpha^2 + b_1\alpha + c_1 = 0$

$a_2\alpha^2 + b_2\alpha + c_2 = 0$ Then

$$\boxed{\therefore\ \alpha = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2} = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}}$$

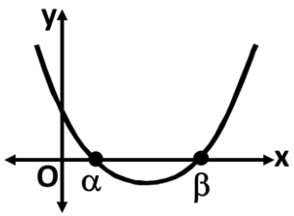
Condition for both the common roots

If both roots of the given equations are common then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Quadratic expression and its graph

Case-I : If a > 0 and D > 0

Then quadratic equation has two roots and graph cuts the x-axis at two distinct points.

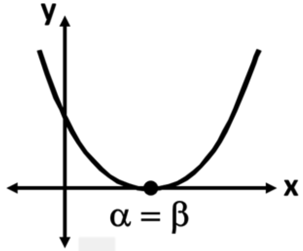


(i) For $\alpha < x < \beta$
$\Rightarrow$ y is negative.

(ii) For $x < \alpha$ or $x > \beta$
$\Rightarrow$ y is positive.

Case-II : If a > 0 and D = 0

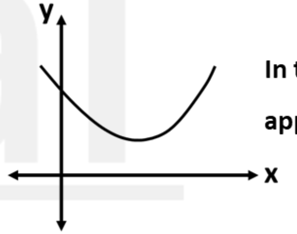
Both zeroes of polynomial coincides, Hence the curve touches x-axis.



$y = a(x-\alpha)^2$ &

$y \geq 0$, for $x \in R$.

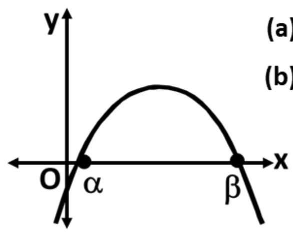
Case-III : If a > 0 and D < 0



In this case imaginary roots appears & $y > 0$ for $x \in R$.

Case-IV : If a < 0 and D > 0

The graph is downward and cuts the x-axis at two distinct points.



(a) $y > 0$, if $\alpha < x < \beta$

(b) $y < 0$, if $x < \alpha$ or $x > \beta$

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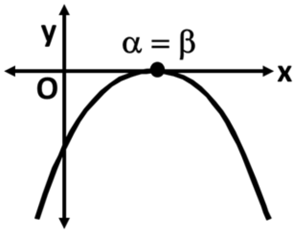
Quadratic Equation

Quadratic expression and its graph

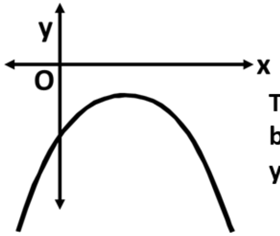
Case-V : If a < 0 and D = 0



The graph touches the x-axis from below.

In this case $y \le 0$ for $x \in R$.

Case-VI : If a < 0 and D < 0



Then graph lies completely below the x-axis and $y < 0$ for $x \in R$.

Important Note

(1) The quadratic expression $ax^2 + bx + c$, where a, b, c $\in$ R is positive $\forall$ x $\in$ R, if a > 0 and D < 0 (Case-III).

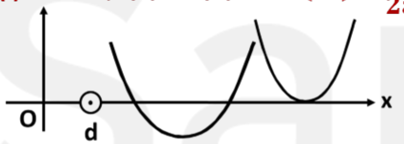
(2) The quadratic expression $ax^2 + bx + c$, where a, b, c $\in$ R is negative $\forall$ x $\in$ R, if a < 0 and D < 0 (Case-VI).

Location of Roots

TYPE-I :

Both roots of the quadratic equation are greater than a specified number say (d).
The necessary and sufficient condition for this are:

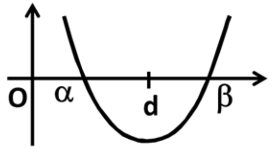
(i) $D \ge 0$; (ii) $a.f(d) > 0$; (iii) $-\dfrac{b}{2a} > d$



TYPE-II :

Both roots lie on either side of a fixed number say (d). Alternatively one root is greater than d and other less than d or d lies between the roots of the given equation.

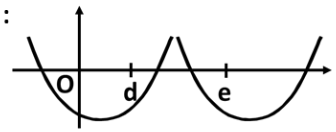
$a.f(d) < 0$



TYPE-III :

Exactly one root lies in the interval (d, e) when d < e.
Conditions for this are :

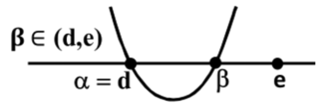
$f(d) \cdot f(e) < 0$

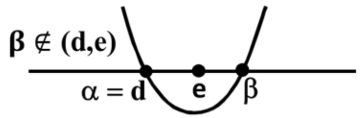


An another case arises when $f(d) \cdot f(e) = 0$, then

For f (d) = 0

One root is d check if other root lies between d and e or not.





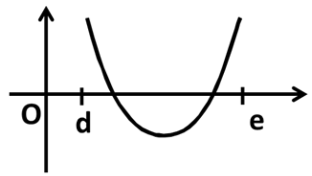
Similarly for f(e) = 0.

TYPE-IV :

When both roots are confined between the number d and e (d < e). Conditions for this are

(i) $D \ge 0$; (ii) $a.f(d) > 0$; (iii) $a.f(e) > 0$

(iv) $d < -\dfrac{b}{2a} < e$

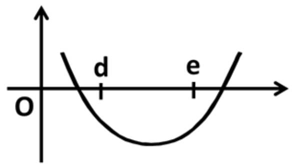


TYPE-V :

One root is greater than e and the other root is less than d. Conditions are (where d < e)

(i) $a.f(d) < 0$

(ii) $a.f(e) < 0$



MIND MAP

Quadratic Equation

Theory of equations

Relation between roots and coefficients of polynomial equation

For Quadratic Equation

If α and β are roots of a quad. eq. $ax^2 + bx + c = 0$ then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

For Cubic Equation

If α, β and γ are roots of a cubic eq. $ax^3 + bx^2 + cx + d = 0$ then

$$\alpha + \beta + \gamma = -\frac{b}{a}; \ \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \ \& \ \alpha\beta\gamma = -\frac{d}{a}$$

For Bi-quadratic Equation

If α, β, γ and δ are roots of a bi-quadratic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ then

$$\alpha + \beta + \gamma + \delta = \frac{-b}{a}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$$

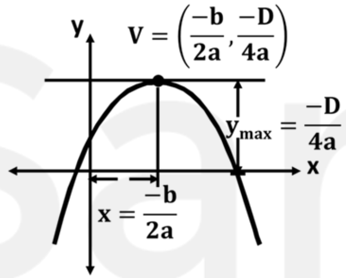
$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = \frac{-d}{a}$$

$$\& \ \alpha\beta\gamma\delta = \frac{e}{a}$$

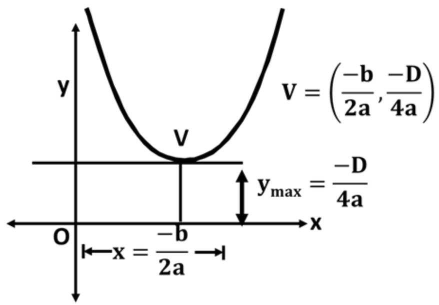
Note Polynomial equations of odd degree with real coefficient must have at least one real root as imaginary roots always occur in pair of conjugates.

Maximum and Minimum values of quadratic and rational functions

If $a < 0$, $y_{max} = \frac{-D}{4a}$ & it occurs at $x = \frac{-b}{2a}$



If $a > 0$, $y_{min} = \frac{-D}{4a}$ & it occurs at $x = \frac{-b}{2a}$



where $D = b^2 - 4ac$

Range of functions expressed in the form of $\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are either linear or quadratic polynomials.

TYPE-I : $y = \frac{ax + b}{px + q} \frac{(\text{Linear})}{(\text{Linear})}$

TYPE-II : $y = \frac{ax + b}{px^2 + qx + r} \frac{(\text{linear})}{(\text{quadratic})}$

TYPE-III : $y = \frac{ax^2 + bx + c}{px^2 + qx + r} \left(\frac{\text{Quadratic}}{\text{Quadratic}}\right)$

TYPE-IV : $y = \frac{ax^2 + bx + c}{px^2 + qx + r} \left(\frac{\text{Quadratic}}{\text{Quadratic}}\right)$

TYPE-V : $y = \frac{ax^2 + px + c}{px^2 + qx + r}$

Resolving a general quadratic exp. in x and y into two linear factors

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + C$

Now f(x, y) can be writing as product of two linear factors only when

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$