



1062CH02

# POLYNOMIALS

# 2

## 2.1 Introduction

In Class IX, you have studied polynomials in one variable and their degrees. Recall that if  $p(x)$  is a polynomial in  $x$ , the highest power of  $x$  in  $p(x)$  is called **the degree of the polynomial**  $p(x)$ . For example,  $4x + 2$  is a polynomial in the variable  $x$  of degree 1,  $2y^2 - 3y + 4$  is a polynomial in the variable  $y$  of degree 2,  $5x^3 - 4x^2 + x - \sqrt{2}$

is a polynomial in the variable  $x$  of degree 3 and  $7u^6 - \frac{3}{2}u^4 + 4u^2 + u - 8$  is a polynomial

in the variable  $u$  of degree 6. Expressions like  $\frac{1}{x-1}$ ,  $\sqrt{x} + 2$ ,  $\frac{1}{x^2 + 2x + 3}$  etc., are not polynomials.

A polynomial of degree 1 is called a **linear polynomial**. For example,  $2x - 3$ ,  $\sqrt{3}x + 5$ ,  $y + \sqrt{2}$ ,  $x - \frac{2}{11}$ ,  $3z + 4$ ,  $\frac{2}{3}u + 1$ , etc., are all linear polynomials. Polynomials such as  $2x + 5 - x^2$ ,  $x^3 + 1$ , etc., are not linear polynomials.

A polynomial of degree 2 is called a **quadratic polynomial**. The name 'quadratic' has been derived from the word 'quadrate', which means 'square'.  $2x^2 + 3x - \frac{2}{5}$ ,

$y^2 - 2$ ,  $2 - x^2 + \sqrt{3}x$ ,  $\frac{u}{3} - 2u^2 + 5$ ,  $\sqrt{5}v^2 - \frac{2}{3}v$ ,  $4z^2 + \frac{1}{7}$  are some examples of quadratic polynomials (whose coefficients are real numbers). More generally, any quadratic polynomial in  $x$  is of the form  $ax^2 + bx + c$ , where  $a, b, c$  are real numbers and  $a \neq 0$ . A polynomial of degree 3 is called a **cubic polynomial**. Some examples of

a cubic polynomial are  $2 - x^3$ ,  $x^3$ ,  $\sqrt{2}x^3$ ,  $3 - x^2 + x^3$ ,  $3x^3 - 2x^2 + x - 1$ . In fact, the most general form of a cubic polynomial is

$$ax^3 + bx^2 + cx + d,$$

where,  $a, b, c, d$  are real numbers and  $a \neq 0$ .

Now consider the polynomial  $p(x) = x^2 - 3x - 4$ . Then, putting  $x = 2$  in the polynomial, we get  $p(2) = 2^2 - 3 \times 2 - 4 = -6$ . The value  $'-6'$ , obtained by replacing  $x$  by 2 in  $x^2 - 3x - 4$ , is the value of  $x^2 - 3x - 4$  at  $x = 2$ . Similarly,  $p(0)$  is the value of  $p(x)$  at  $x = 0$ , which is  $-4$ .

If  $p(x)$  is a polynomial in  $x$ , and if  $k$  is any real number, then the value obtained by replacing  $x$  by  $k$  in  $p(x)$ , is called **the value of  $p(x)$  at  $x = k$** , and is denoted by  $p(k)$ .

What is the value of  $p(x) = x^2 - 3x - 4$  at  $x = -1$ ? We have :

$$p(-1) = (-1)^2 - \{3 \times (-1)\} - 4 = 0$$

Also, note that  $p(4) = 4^2 - (3 \times 4) - 4 = 0$ .

As  $p(-1) = 0$  and  $p(4) = 0$ ,  $-1$  and  $4$  are called the zeroes of the quadratic polynomial  $x^2 - 3x - 4$ . More generally, a real number  $k$  is said to be a **zero of a polynomial  $p(x)$** , if  $p(k) = 0$ .

You have already studied in Class IX, how to find the zeroes of a linear polynomial. For example, if  $k$  is a zero of  $p(x) = 2x + 3$ , then  $p(k) = 0$  gives us  $2k + 3 = 0$ , i.e.,  $k = -\frac{3}{2}$ .

In general, if  $k$  is a zero of  $p(x) = ax + b$ , then  $p(k) = ak + b = 0$ , i.e.,  $k = \frac{-b}{a}$ .

So, the zero of the linear polynomial  $ax + b$  is  $\frac{-b}{a} = \frac{-(\text{Constant term})}{\text{Coefficient of } x}$ .

Thus, the zero of a linear polynomial is related to its coefficients. Does this happen in the case of other polynomials too? For example, are the zeroes of a quadratic polynomial also related to its coefficients?

In this chapter, we will try to answer these questions. We will also study the division algorithm for polynomials.

## 2.2 Geometrical Meaning of the Zeroes of a Polynomial

You know that a real number  $k$  is a zero of the polynomial  $p(x)$  if  $p(k) = 0$ . But why are the zeroes of a polynomial so important? To answer this, first we will see the **geometrical** representations of linear and quadratic polynomials and the geometrical meaning of their zeroes.

Consider first a linear polynomial  $ax + b$ ,  $a \neq 0$ . You have studied in Class IX that the graph of  $y = ax + b$  is a straight line. For example, the graph of  $y = 2x + 3$  is a straight line passing through the points  $(-2, -1)$  and  $(2, 7)$ .

$x$	$-2$	$2$
$y = 2x + 3$	$-1$	$7$

From Fig. 2.1, you can see that the graph of  $y = 2x + 3$  intersects the  $x$ -axis mid-way between  $x = -1$  and  $x = -2$ , that is, at the point  $\left(-\frac{3}{2}, 0\right)$ .

You also know that the zero of  $2x + 3$  is  $-\frac{3}{2}$ . Thus, the zero of the polynomial  $2x + 3$  is the  $x$ -coordinate of the point where the graph of  $y = 2x + 3$  intersects the  $x$ -axis.

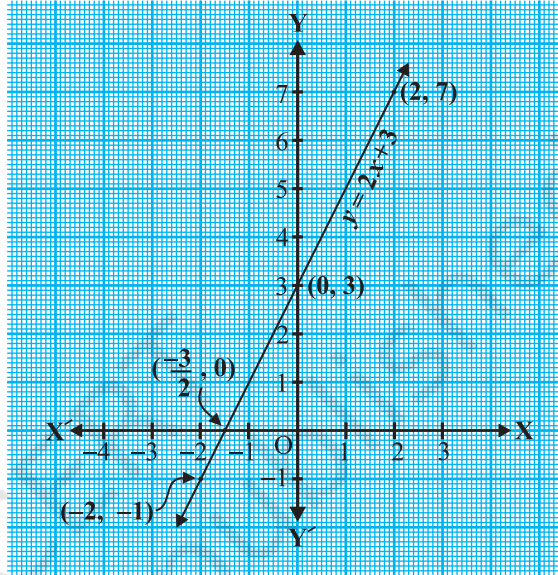


Fig. 2.1

In general, for a linear polynomial  $ax + b$ ,  $a \neq 0$ , the graph of  $y = ax + b$  is a straight line which intersects the  $x$ -axis at exactly one point, namely,  $\left(\frac{-b}{a}, 0\right)$ .

Therefore, the linear polynomial  $ax + b$ ,  $a \neq 0$ , has exactly one zero, namely, the  $x$ -coordinate of the point where the graph of  $y = ax + b$  intersects the  $x$ -axis.

Now, let us look for the geometrical meaning of a zero of a quadratic polynomial. Consider the quadratic polynomial  $x^2 - 3x - 4$ . Let us see what the graph\* of  $y = x^2 - 3x - 4$  looks like. Let us list a few values of  $y = x^2 - 3x - 4$  corresponding to a few values for  $x$  as given in Table 2.1.

\* Plotting of graphs of quadratic or cubic polynomials is not meant to be done by the students, nor is to be evaluated.

Table 2.1

$x$	-2	-1	0	1	2	3	4	5
$y = x^2 - 3x - 4$	6	0	-4	-6	-6	-4	0	6

If we locate the points listed above on a graph paper and draw the graph, it will actually look like the one given in Fig. 2.2.

In fact, for any quadratic polynomial  $ax^2 + bx + c$ ,  $a \neq 0$ , the graph of the corresponding equation  $y = ax^2 + bx + c$  has one of the two shapes either open upwards like  $\cup$  or open downwards like  $\cap$  depending on whether  $a > 0$  or  $a < 0$ . (These curves are called **parabolas**.)

You can see from Table 2.1 that -1 and 4 are zeroes of the quadratic polynomial. Also note from Fig. 2.2 that -1 and 4 are the  $x$ -coordinates of the points where the graph of  $y = x^2 - 3x - 4$  intersects the  $x$ -axis. Thus, the zeroes of the quadratic polynomial  $x^2 - 3x - 4$  are  $x$ -coordinates of the points where the graph of  $y = x^2 - 3x - 4$  intersects the  $x$ -axis.

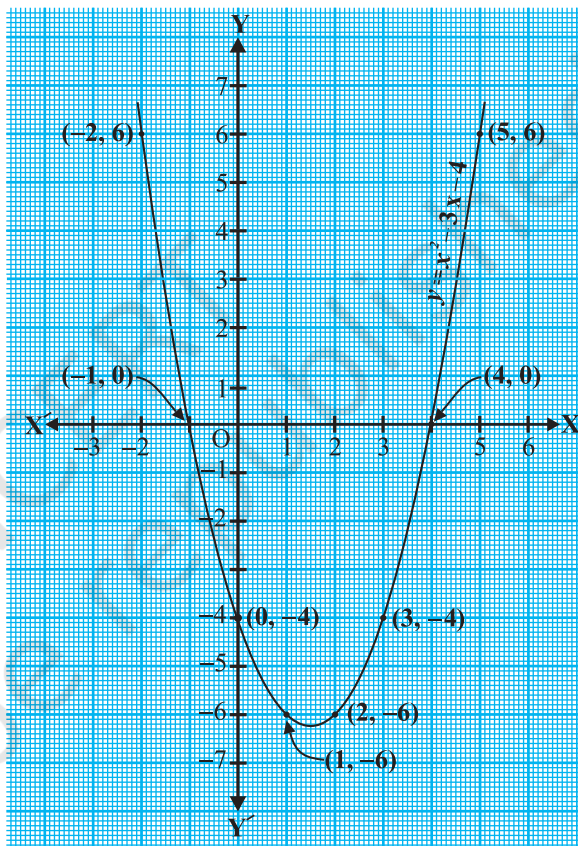


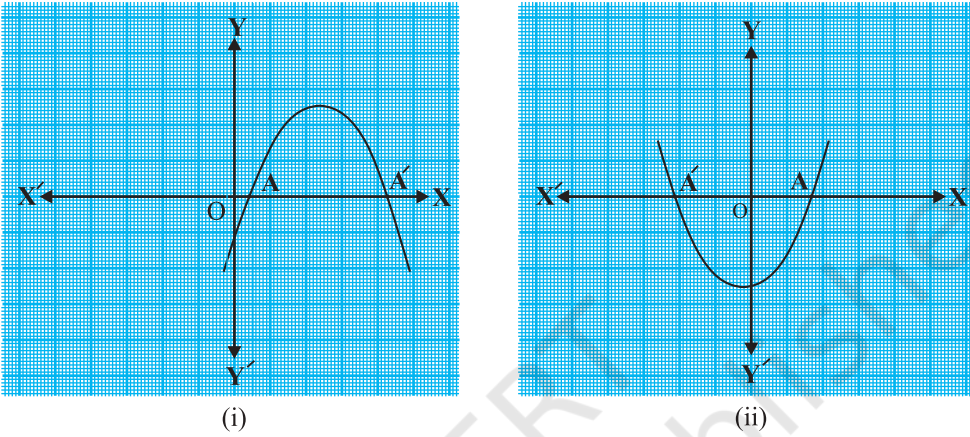
Fig. 2.2

This fact is true for any quadratic polynomial, i.e., the zeroes of a quadratic polynomial  $ax^2 + bx + c$ ,  $a \neq 0$ , are precisely the  $x$ -coordinates of the points where the parabola representing  $y = ax^2 + bx + c$  intersects the  $x$ -axis.

From our observation earlier about the shape of the graph of  $y = ax^2 + bx + c$ , the following three cases can happen:

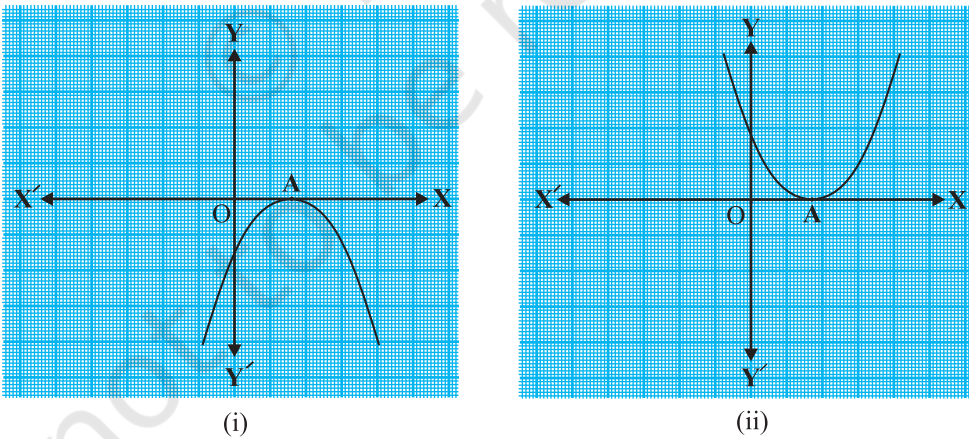
**Case (i) :** Here, the graph cuts  $x$ -axis at two distinct points  $A$  and  $A'$ .

The  $x$ -coordinates of  $A$  and  $A'$  are the **two zeroes** of the quadratic polynomial  $ax^2 + bx + c$  in this case (see Fig. 2.3).



**Fig. 2.3**

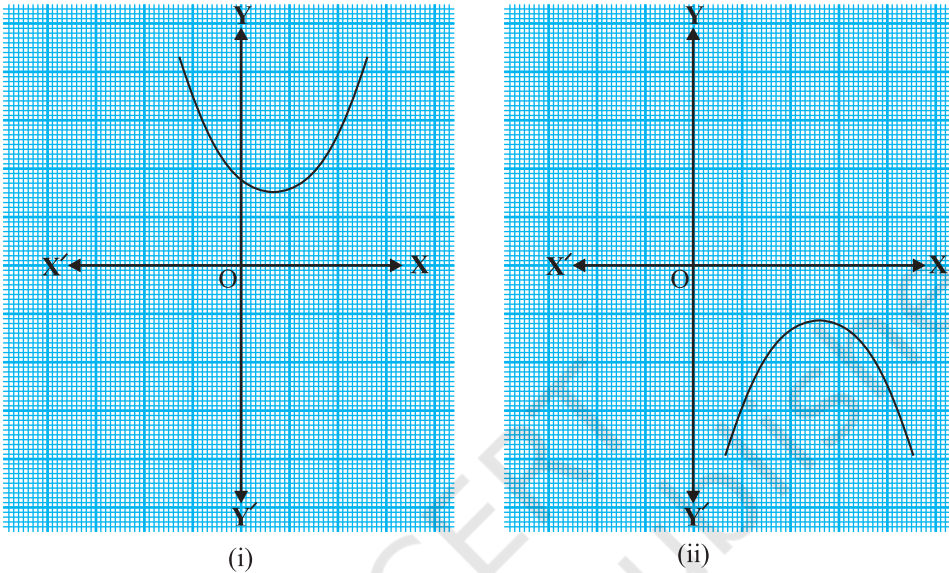
**Case (ii) :** Here, the graph cuts the  $x$ -axis at exactly one point, i.e., at two coincident points. So, the two points  $A$  and  $A'$  of Case (i) coincide here to become one point  $A$  (see Fig. 2.4).



**Fig. 2.4**

The  $x$ -coordinate of  $A$  is the **only zero** for the quadratic polynomial  $ax^2 + bx + c$  in this case.

**Case (iii) :** Here, the graph is either completely above the  $x$ -axis or completely below the  $x$ -axis. So, it does not cut the  $x$ -axis at any point (see Fig. 2.5).



**Fig. 2.5**

So, the quadratic polynomial  $ax^2 + bx + c$  has **no zero** in this case.

So, you can see geometrically that a quadratic polynomial can have either two distinct zeroes or two equal zeroes (i.e., one zero), or no zero. This also means that a polynomial of degree 2 has at most two zeroes.

Now, what do you expect the geometrical meaning of the zeroes of a cubic polynomial to be? Let us find out. Consider the cubic polynomial  $x^3 - 4x$ . To see what the graph of  $y = x^3 - 4x$  looks like, let us list a few values of  $y$  corresponding to a few values for  $x$  as shown in Table 2.2.

**Table 2.2**

$x$	-2	-1	0	1	2
$y = x^3 - 4x$	0	3	0	-3	0

Locating the points of the table on a graph paper and drawing the graph, we see that the graph of  $y = x^3 - 4x$  actually looks like the one given in Fig. 2.6.

We see from the table above that  $-2$ ,  $0$  and  $2$  are zeroes of the cubic polynomial  $x^3 - 4x$ . Observe that  $-2$ ,  $0$  and  $2$  are, in fact, the  $x$ -coordinates of the only points where the graph of  $y = x^3 - 4x$  intersects the  $x$ -axis. Since the curve meets the  $x$ -axis in only these 3 points, their  $x$ -coordinates are the only zeroes of the polynomial.

Let us take a few more examples. Consider the cubic polynomials  $x^3$  and  $x^3 - x^2$ . We draw the graphs of  $y = x^3$  and  $y = x^3 - x^2$  in Fig. 2.7 and Fig. 2.8 respectively.

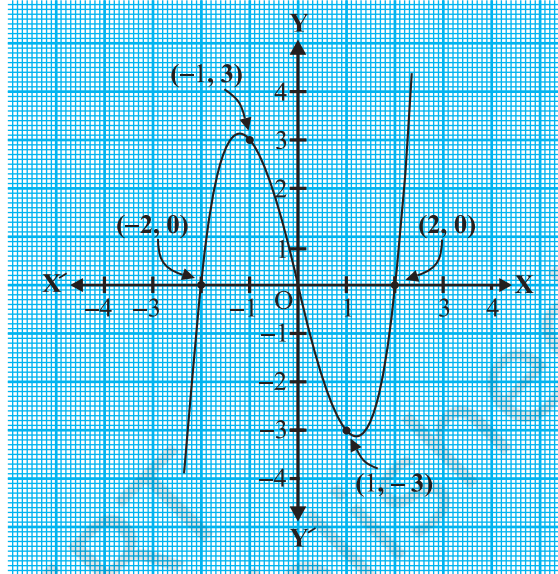


Fig. 2.6

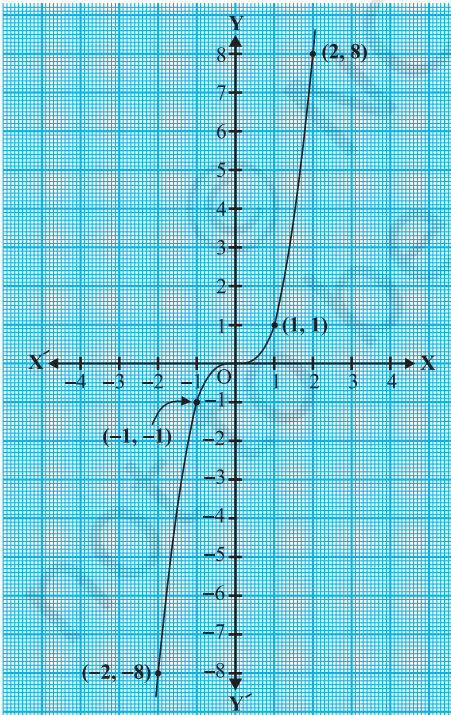


Fig. 2.7

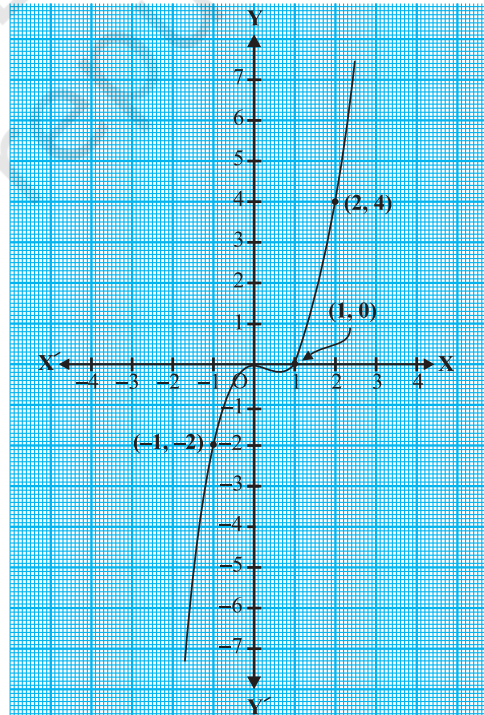


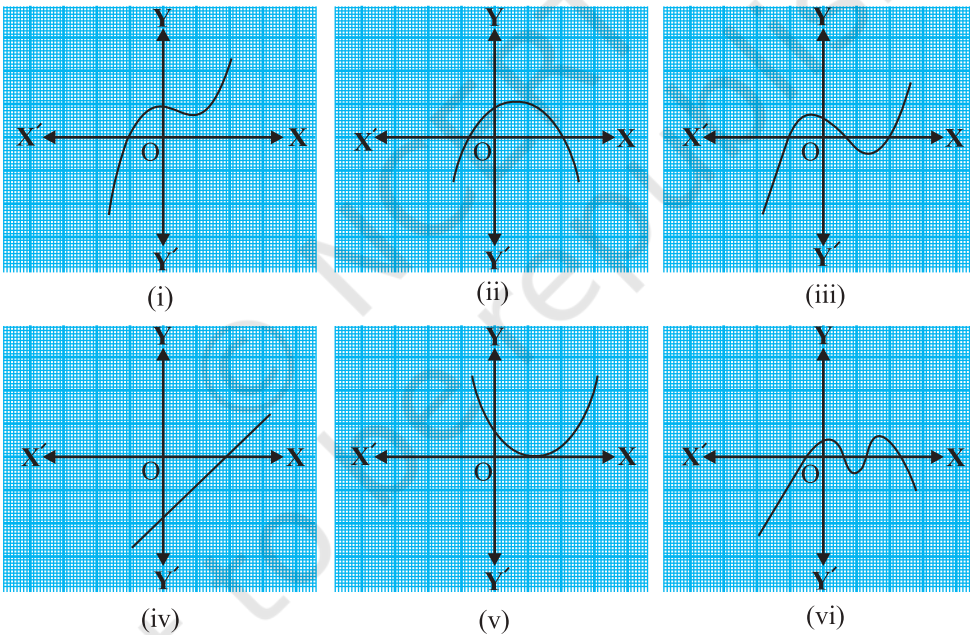
Fig. 2.8

Note that 0 is the only zero of the polynomial  $x^3$ . Also, from Fig. 2.7, you can see that 0 is the  $x$ -coordinate of the only point where the graph of  $y = x^3$  intersects the  $x$ -axis. Similarly, since  $x^3 - x^2 = x^2(x - 1)$ , 0 and 1 are the only zeroes of the polynomial  $x^3 - x^2$ . Also, from Fig. 2.8, these values are the  $x$ -coordinates of the only points where the graph of  $y = x^3 - x^2$  intersects the  $x$ -axis.

From the examples above, we see that there are at most 3 zeroes for any cubic polynomial. In other words, any polynomial of degree 3 can have at most three zeroes.

**Remark :** In general, given a polynomial  $p(x)$  of degree  $n$ , the graph of  $y = p(x)$  intersects the  $x$ -axis at at most  $n$  points. Therefore, a polynomial  $p(x)$  of degree  $n$  has at most  $n$  zeroes.

**Example 1 :** Look at the graphs in Fig. 2.9 given below. Each is the graph of  $y = p(x)$ , where  $p(x)$  is a polynomial. For each of the graphs, find the number of zeroes of  $p(x)$ .



**Fig. 2.9**

**Solution :**

- (i) The number of zeroes is 1 as the graph intersects the  $x$ -axis at one point only.
- (ii) The number of zeroes is 2 as the graph intersects the  $x$ -axis at two points.
- (iii) The number of zeroes is 3. (Why?)



- (iv) The number of zeroes is 1. (Why?)  
 (v) The number of zeroes is 1. (Why?)  
 (vi) The number of zeroes is 4. (Why?)

### EXERCISE 2.1

1. The graphs of  $y = p(x)$  are given in Fig. 2.10 below, for some polynomials  $p(x)$ . Find the number of zeroes of  $p(x)$ , in each case.

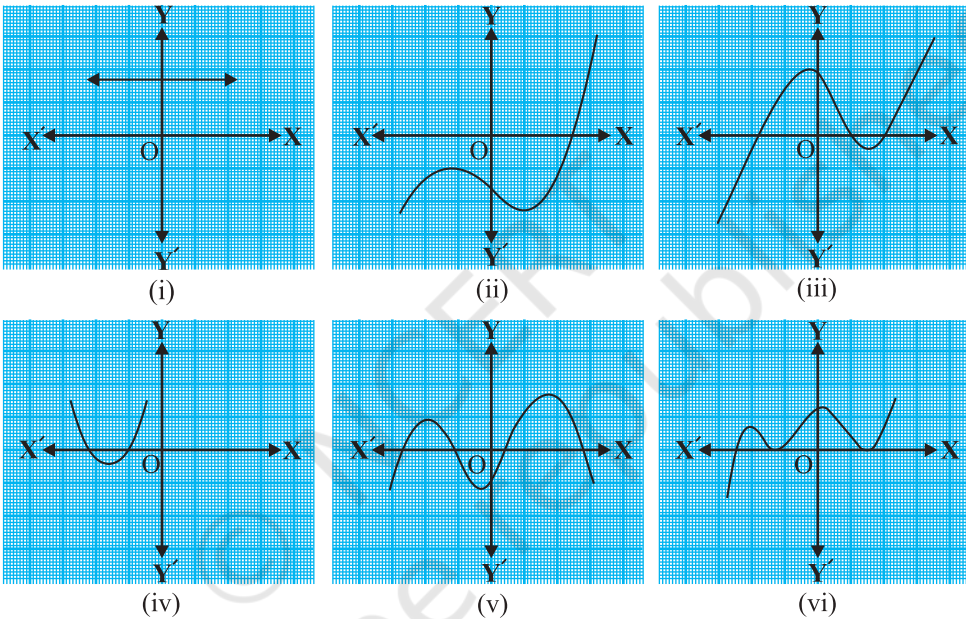


Fig. 2.10

### 2.3 Relationship between Zeroes and Coefficients of a Polynomial

You have already seen that zero of a linear polynomial  $ax + b$  is  $-\frac{b}{a}$ . We will now try to answer the question raised in Section 2.1 regarding the relationship between zeroes and coefficients of a quadratic polynomial. For this, let us take a quadratic polynomial, say  $p(x) = 2x^2 - 8x + 6$ . In Class IX, you have learnt how to factorise quadratic polynomials by splitting the middle term. So, here we need to split the middle term  $-8x$  as a sum of two terms, whose product is  $6 \times 2x^2 = 12x^2$ . So, we write

$$\begin{aligned} 2x^2 - 8x + 6 &= 2x^2 - 6x - 2x + 6 = 2x(x - 3) - 2(x - 3) \\ &= (2x - 2)(x - 3) = 2(x - 1)(x - 3) \end{aligned}$$

So, the value of  $p(x) = 2x^2 - 8x + 6$  is zero when  $x - 1 = 0$  or  $x - 3 = 0$ , i.e., when  $x = 1$  or  $x = 3$ . So, the zeroes of  $2x^2 - 8x + 6$  are 1 and 3. Observe that :

$$\text{Sum of its zeroes} = 1 + 3 = 4 = \frac{-(-8)}{2} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$$

$$\text{Product of its zeroes} = 1 \times 3 = 3 = \frac{6}{2} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Let us take one more quadratic polynomial, say,  $p(x) = 3x^2 + 5x - 2$ . By the method of splitting the middle term,

$$\begin{aligned} 3x^2 + 5x - 2 &= 3x^2 + 6x - x - 2 = 3x(x + 2) - 1(x + 2) \\ &= (3x - 1)(x + 2) \end{aligned}$$

Hence, the value of  $3x^2 + 5x - 2$  is zero when either  $3x - 1 = 0$  or  $x + 2 = 0$ , i.e., when  $x = \frac{1}{3}$  or  $x = -2$ . So, the zeroes of  $3x^2 + 5x - 2$  are  $\frac{1}{3}$  and  $-2$ . Observe that :

$$\text{Sum of its zeroes} = \frac{1}{3} + (-2) = \frac{-5}{3} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2}$$

$$\text{Product of its zeroes} = \frac{1}{3} \times (-2) = \frac{-2}{3} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

In general, if  $\alpha^*$  and  $\beta^*$  are the zeroes of the quadratic polynomial  $p(x) = ax^2 + bx + c$ ,  $a \neq 0$ , then you know that  $x - \alpha$  and  $x - \beta$  are the factors of  $p(x)$ . Therefore,

$$\begin{aligned} ax^2 + bx + c &= k(x - \alpha)(x - \beta), \text{ where } k \text{ is a constant} \\ &= k[x^2 - (\alpha + \beta)x + \alpha\beta] \\ &= kx^2 - k(\alpha + \beta)x + k\alpha\beta \end{aligned}$$

Comparing the coefficients of  $x^2$ ,  $x$  and constant terms on both the sides, we get

$$a = k, b = -k(\alpha + \beta) \text{ and } c = k\alpha\beta.$$

This gives 
$$\alpha + \beta = \frac{-b}{a},$$

$$\alpha\beta = \frac{c}{a}$$

\*  $\alpha, \beta$  are Greek letters pronounced as 'alpha' and 'beta' respectively. We will use later one more letter ' $\gamma$ ' pronounced as 'gamma'.

i.e., 
$$\text{sum of zeroes} = \alpha + \beta = -\frac{b}{a} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2},$$

$$\text{product of zeroes} = \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

Let us consider some examples.

**Example 2 :** Find the zeroes of the quadratic polynomial  $x^2 + 7x + 10$ , and verify the relationship between the zeroes and the coefficients.

**Solution :** We have

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

So, the value of  $x^2 + 7x + 10$  is zero when  $x + 2 = 0$  or  $x + 5 = 0$ , i.e., when  $x = -2$  or  $x = -5$ . Therefore, the zeroes of  $x^2 + 7x + 10$  are  $-2$  and  $-5$ . Now,

$$\text{sum of zeroes} = -2 + (-5) = -7 = \frac{-7}{1} = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2},$$

$$\text{product of zeroes} = (-2) \times (-5) = 10 = \frac{10}{1} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

**Example 3 :** Find the zeroes of the polynomial  $x^2 - 3$  and verify the relationship between the zeroes and the coefficients.

**Solution :** Recall the identity  $a^2 - b^2 = (a - b)(a + b)$ . Using it, we can write:

$$x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$$

So, the value of  $x^2 - 3$  is zero when  $x = \sqrt{3}$  or  $x = -\sqrt{3}$ .

Therefore, the zeroes of  $x^2 - 3$  are  $\sqrt{3}$  and  $-\sqrt{3}$ .

Now,

$$\text{sum of zeroes} = \sqrt{3} - \sqrt{3} = 0 = \frac{-(\text{Coefficient of } x)}{\text{Coefficient of } x^2},$$

$$\text{product of zeroes} = (\sqrt{3})(-\sqrt{3}) = -3 = \frac{-3}{1} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

**Example 4 :** Find a quadratic polynomial, the sum and product of whose zeroes are  $-3$  and  $2$ , respectively.

**Solution :** Let the quadratic polynomial be  $ax^2 + bx + c$ , and its zeroes be  $\alpha$  and  $\beta$ . We have

$$\alpha + \beta = -3 = \frac{-b}{a},$$

and 
$$\alpha\beta = 2 = \frac{c}{a}.$$

If  $a = 1$ , then  $b = 3$  and  $c = 2$ .

So, one quadratic polynomial which fits the given conditions is  $x^2 + 3x + 2$ .

You can check that any other quadratic polynomial that fits these conditions will be of the form  $k(x^2 + 3x + 2)$ , where  $k$  is real.

Let us now look at cubic polynomials. Do you think a similar relation holds between the zeroes of a cubic polynomial and its coefficients?

Let us consider  $p(x) = 2x^3 - 5x^2 - 14x + 8$ .

You can check that  $p(x) = 0$  for  $x = 4, -2, \frac{1}{2}$ . Since  $p(x)$  can have at most three zeroes, these are the zeroes of  $2x^3 - 5x^2 - 14x + 8$ . Now,

$$\text{sum of the zeroes} = 4 + (-2) + \frac{1}{2} = \frac{5}{2} = \frac{-(-5)}{2} = \frac{-(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3},$$

$$\text{product of the zeroes} = 4 \times (-2) \times \frac{1}{2} = -4 = \frac{-8}{2} = \frac{-\text{Constant term}}{\text{Coefficient of } x^3}.$$

However, there is one more relationship here. Consider the sum of the products of the zeroes taken two at a time. We have

$$\begin{aligned} & \{4 \times (-2)\} + \left\{(-2) \times \frac{1}{2}\right\} + \left\{\frac{1}{2} \times 4\right\} \\ &= -8 - 1 + 2 = -7 = \frac{-14}{2} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}. \end{aligned}$$

In general, it can be proved that if  $\alpha, \beta, \gamma$  are the zeroes of the cubic polynomial  $ax^3 + bx^2 + cx + d$ , then

$$\alpha + \beta + \gamma = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

$$\alpha\beta\gamma = \frac{-d}{a}.$$

Let us consider an example.

**Example 5\*** : Verify that 3, -1,  $-\frac{1}{3}$  are the zeroes of the cubic polynomial  $p(x) = 3x^3 - 5x^2 - 11x - 3$ , and then verify the relationship between the zeroes and the coefficients.

**Solution** : Comparing the given polynomial with  $ax^3 + bx^2 + cx + d$ , we get

$a = 3, b = -5, c = -11, d = -3$ . Further

$$p(3) = 3 \times 3^3 - (5 \times 3^2) - (11 \times 3) - 3 = 81 - 45 - 33 - 3 = 0,$$

$$p(-1) = 3 \times (-1)^3 - 5 \times (-1)^2 - 11 \times (-1) - 3 = -3 - 5 + 11 - 3 = 0,$$

$$p\left(-\frac{1}{3}\right) = 3 \times \left(-\frac{1}{3}\right)^3 - 5 \times \left(-\frac{1}{3}\right)^2 - 11 \times \left(-\frac{1}{3}\right) - 3,$$

$$= -\frac{1}{9} - \frac{5}{9} + \frac{11}{3} - 3 = -\frac{2}{3} + \frac{2}{3} = 0$$

Therefore, 3, -1 and  $-\frac{1}{3}$  are the zeroes of  $3x^3 - 5x^2 - 11x - 3$ .

So, we take  $\alpha = 3, \beta = -1$  and  $\gamma = -\frac{1}{3}$ .

Now,

$$\alpha + \beta + \gamma = 3 + (-1) + \left(-\frac{1}{3}\right) = 2 - \frac{1}{3} = \frac{5}{3} = \frac{-(-5)}{3} = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3 \times (-1) + (-1) \times \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) \times 3 = -3 + \frac{1}{3} - 1 = \frac{-11}{3} = \frac{c}{a},$$

$$\alpha\beta\gamma = 3 \times (-1) \times \left(-\frac{1}{3}\right) = 1 = \frac{-(-3)}{3} = \frac{-d}{a}.$$

\* Not from the examination point of view.

## EXERCISE 2.2

1. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

(i)  $x^2 - 2x - 8$

(ii)  $4s^2 - 4s + 1$

(iii)  $6x^2 - 3 - 7x$

(iv)  $4u^2 + 8u$

(v)  $t^2 - 15$

(vi)  $3x^2 - x - 4$

2. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.

(i)  $\frac{1}{4}, -1$

(ii)  $\sqrt{2}, \frac{1}{3}$

(iii)  $0, \sqrt{5}$

(iv)  $1, 1$

(v)  $-\frac{1}{4}, \frac{1}{4}$

(vi)  $4, 1$

## 2.4 Division Algorithm for Polynomials

You know that a cubic polynomial has at most three zeroes. However, if you are given only one zero, can you find the other two? For this, let us consider the cubic polynomial  $x^3 - 3x^2 - x + 3$ . If we tell you that one of its zeroes is 1, then you know that  $x - 1$  is a factor of  $x^3 - 3x^2 - x + 3$ . So, you can divide  $x^3 - 3x^2 - x + 3$  by  $x - 1$ , as you have learnt in Class IX, to get the quotient  $x^2 - 2x - 3$ .

Next, you could get the factors of  $x^2 - 2x - 3$ , by splitting the middle term, as  $(x + 1)(x - 3)$ . This would give you

$$\begin{aligned} x^3 - 3x^2 - x + 3 &= (x - 1)(x^2 - 2x - 3) \\ &= (x - 1)(x + 1)(x - 3) \end{aligned}$$

So, all the three zeroes of the cubic polynomial are now known to you as 1, -1, 3.

Let us discuss the method of dividing one polynomial by another in some detail. Before noting the steps formally, consider an example.

**Example 6 :** Divide  $2x^2 + 3x + 1$  by  $x + 2$ .

**Solution :** Note that we stop the division process when either the remainder is zero or its degree is less than the degree of the divisor. So, here the quotient is  $2x - 1$  and the remainder is 3. Also,

$$\begin{aligned} (2x - 1)(x + 2) + 3 &= 2x^2 + 3x - 2 + 3 = 2x^2 + 3x + 1 \\ \text{i.e., } 2x^2 + 3x + 1 &= (x + 2)(2x - 1) + 3 \end{aligned}$$

Therefore, Dividend = Divisor  $\times$  Quotient + Remainder

Let us now extend this process to divide a polynomial by a quadratic polynomial.

$$\begin{array}{r} \phantom{x + 2} \overline{) 2x^2 + 3x + 1} \\ \underline{2x^2 + 4x} \phantom{+ 1} \\ \phantom{2x^2 + } -x + 1 \\ \phantom{2x^2 + } \underline{-x - 2} \\ \phantom{2x^2 + } \phantom{-x} 3 \end{array}$$



**Solution :** Note that the given polynomials are not in standard form. To carry out division, we first write both the dividend and divisor in decreasing orders of their degrees.

So, dividend =  $-x^3 + 3x^2 - 3x + 5$  and divisor =  $-x^2 + x - 1$ .

Division process is shown on the right side.

We stop here since degree (3) = 0 < 2 = degree ( $-x^2 + x - 1$ ).

So, quotient =  $x - 2$ , remainder = 3.

Now,

Divisor  $\times$  Quotient + Remainder

$$\begin{aligned} &= (-x^2 + x - 1)(x - 2) + 3 \\ &= -x^3 + x^2 - x + 2x^2 - 2x + 2 + 3 \\ &= -x^3 + 3x^2 - 3x + 5 \\ &= \text{Dividend} \end{aligned}$$

In this way, the division algorithm is verified.

**Example 9 :** Find all the zeroes of  $2x^4 - 3x^3 - 3x^2 + 6x - 2$ , if you know that two of its zeroes are  $\sqrt{2}$  and  $-\sqrt{2}$ .

**Solution :** Since two zeroes are  $\sqrt{2}$  and  $-\sqrt{2}$ ,  $(x - \sqrt{2})(x + \sqrt{2}) = x^2 - 2$  is a factor of the given polynomial. Now, we divide the given polynomial by  $x^2 - 2$ .

$$\begin{array}{r} 2x^2 - 3x + 1 \\ x^2 - 2 \overline{) 2x^4 - 3x^3 - 3x^2 + 6x - 2} \\ \underline{- 2x^4 \phantom{- 3x^3} - 4x^2 \phantom{+ 6x} - 2} \\ -3x^3 + x^2 + 6x - 2 \\ \underline{-3x^3 \phantom{+ x^2} + 6x} \\ x^2 - 2 \\ \underline{x^2 \phantom{- 2} - 2} \\ - \phantom{x^2} + \\ \hline 0 \end{array}$$

First term of quotient is  $\frac{2x^4}{x^2} = 2x^2$

Second term of quotient is  $\frac{-3x^3}{x^2} = -3x$

Third term of quotient is  $\frac{x^2}{x^2} = 1$

$$\begin{array}{r} x - 2 \\ -x^2 + x - 1 \overline{) -x^3 + 3x^2 - 3x + 5} \\ \underline{-x^3 + x^2 - x} \\ 2x^2 - 2x + 5 \\ \underline{2x^2 - 2x + 2} \\ 3 \end{array}$$



So,  $2x^4 - 3x^3 - 3x^2 + 6x - 2 = (x^2 - 2)(2x^2 - 3x + 1)$ .

Now, by splitting  $-3x$ , we factorise  $2x^2 - 3x + 1$  as  $(2x - 1)(x - 1)$ . So, its zeroes are given by  $x = \frac{1}{2}$  and  $x = 1$ . Therefore, the zeroes of the given polynomial are

$\sqrt{2}$ ,  $-\sqrt{2}$ ,  $\frac{1}{2}$ , and  $1$ .

### EXERCISE 2.3

- Divide the polynomial  $p(x)$  by the polynomial  $g(x)$  and find the quotient and remainder in each of the following :
  - $p(x) = x^3 - 3x^2 + 5x - 3$ ,  $g(x) = x^2 - 2$
  - $p(x) = x^4 - 3x^2 + 4x + 5$ ,  $g(x) = x^2 + 1 - x$
  - $p(x) = x^4 - 5x + 6$ ,  $g(x) = 2 - x^2$
- Check whether the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial:
  - $t^2 - 3$ ,  $2t^4 + 3t^3 - 2t^2 - 9t - 12$
  - $x^2 + 3x + 1$ ,  $3x^4 + 5x^3 - 7x^2 + 2x + 2$
  - $x^3 - 3x + 1$ ,  $x^5 - 4x^3 + x^2 + 3x + 1$
- Obtain all other zeroes of  $3x^4 + 6x^3 - 2x^2 - 10x - 5$ , if two of its zeroes are  $\sqrt{\frac{5}{3}}$  and  $-\sqrt{\frac{5}{3}}$ .
- On dividing  $x^3 - 3x^2 + x + 2$  by a polynomial  $g(x)$ , the quotient and remainder were  $x - 2$  and  $-2x + 4$ , respectively. Find  $g(x)$ .
- Give examples of polynomials  $p(x)$ ,  $g(x)$ ,  $q(x)$  and  $r(x)$ , which satisfy the division algorithm and
  - $\deg p(x) = \deg q(x)$
  - $\deg q(x) = \deg r(x)$
  - $\deg r(x) = 0$

### EXERCISE 2.4 (Optional)\*

- Verify that the numbers given alongside of the cubic polynomials below are their zeroes. Also verify the relationship between the zeroes and the coefficients in each case:
  - $2x^3 + x^2 - 5x + 2$ ;  $\frac{1}{2}$ ,  $1$ ,  $-2$
  - $x^3 - 4x^2 + 5x - 2$ ;  $2$ ,  $1$ ,  $1$
- Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as  $2$ ,  $-7$ ,  $-14$  respectively.

\*These exercises are not from the examination point of view.

3. If the zeroes of the polynomial  $x^3 - 3x^2 + x + 1$  are  $a - b$ ,  $a$ ,  $a + b$ , find  $a$  and  $b$ .
4. If two zeroes of the polynomial  $x^4 - 6x^3 - 26x^2 + 138x - 35$  are  $2 \pm \sqrt{3}$ , find other zeroes.
5. If the polynomial  $x^4 - 6x^3 + 16x^2 - 25x + 10$  is divided by another polynomial  $x^2 - 2x + k$ , the remainder comes out to be  $x + a$ , find  $k$  and  $a$ .

## 2.5 Summary

In this chapter, you have studied the following points:

1. Polynomials of degrees 1, 2 and 3 are called linear, quadratic and cubic polynomials respectively.
2. A quadratic polynomial in  $x$  with real coefficients is of the form  $ax^2 + bx + c$ , where  $a, b, c$  are real numbers with  $a \neq 0$ .
3. The zeroes of a polynomial  $p(x)$  are precisely the  $x$ -coordinates of the points, where the graph of  $y = p(x)$  intersects the  $x$ -axis.
4. A quadratic polynomial can have at most 2 zeroes and a cubic polynomial can have at most 3 zeroes.
5. If  $\alpha$  and  $\beta$  are the zeroes of the quadratic polynomial  $ax^2 + bx + c$ , then

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

6. If  $\alpha, \beta, \gamma$  are the zeroes of the cubic polynomial  $ax^3 + bx^2 + cx + d$ , then

$$\alpha + \beta + \gamma = -\frac{b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

$$\text{and} \quad \alpha\beta\gamma = -\frac{d}{a}.$$

7. The division algorithm states that given any polynomial  $p(x)$  and any non-zero polynomial  $g(x)$ , there are polynomials  $q(x)$  and  $r(x)$  such that

$$p(x) = g(x)q(x) + r(x),$$

where  $r(x) = 0$  or  $\text{degree } r(x) < \text{degree } g(x)$ .