Exercise 11.3

Question 1:

In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a)z = 2 (b) $x+y+z=1$

(c) $2x+3y-z=5$ (d)5y + 8 = 0

Answer

(a) The equation of the plane is z = 2 or 0x + 0y + z = 2 … (1)

The direction ratios of normal are 0, 0, and 1.

$\therefore \sqrt{0^2+0^2+1^2}=1$

Dividing both sides of equation (1) by 1, we obtain

$0.x+0.y+1.z=2$

This is of the form lx + my + nz = d, where l, m, n are the direction cosines of normal to the plane and d is the distance of the perpendicular drawn from the origin.

Therefore, the direction cosines are 0, 0, and 1 and the distance of the plane from the origin is 2 units.

(b) x + y + z = 1 … (1)

The direction ratios of normal are 1, 1, and 1.

$\therefore \sqrt{(1)^2+(1)^2+(1)^2}=\sqrt{3}$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{3}}z=\frac{1}{\sqrt{3}} \qquad ...(2)$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Therefore, the direction cosines of the normal are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \text{ and } \frac{1}{\sqrt{3}}$ and the distance of normal from the origin is $\frac{1}{\sqrt{3}}$ units.

(c) $2x + 3y - z = 5$ … (1)

The direction ratios of normal are 2, 3, and −1.

$$\therefore \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$$

Dividing both sides of equation (1) by $\sqrt{14}$, we obtain

$$\frac{2}{\sqrt{14}}x + \frac{3}{\sqrt{14}}y - \frac{1}{\sqrt{14}}z = \frac{5}{\sqrt{14}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \text{ and } \frac{-1}{\sqrt{14}}$ and the distance of normal from the origin is $\frac{5}{\sqrt{14}}$ units.

(d) $5y + 8 = 0$

$\Rightarrow 0x - 5y + 0z = 8$ … (1)

The direction ratios of normal are 0, −5, and 0.

$$\therefore \sqrt{0 + (-5)^2 + 0} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$-y = \frac{8}{5}$$

This equation is of the form *lx* + *my* + *nz* = *d*, where *l*, *m*, *n* are the direction cosines of normal to the plane and *d* is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are 0, −1, and 0 and the distance of normal from the origin is $\frac{8}{5}$ units.

Question 2:

Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i}+5\hat{j}-6\hat{k}$.

Answer

The normal vector is, $\vec{n}=3\hat{i}+5\hat{j}-6\hat{k}$

$$\therefore \hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{(3)^2+(5)^2+(6)^2}}=\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}$$

It is known that the equation of the plane with position vector $\vec{r}$ is given by, $\vec{r}\cdot\hat{n}=d$

$$\Rightarrow \hat{r}\cdot\left(\frac{3\hat{i}+5\hat{j}-6\hat{k}}{\sqrt{70}}\right)=7$$

This is the vector equation of the required plane.

Question 3:

Find the Cartesian equation of the following planes:

(a) $\vec{r}\cdot\left(\hat{i}+\hat{j}-\hat{k}\right)=2$ (b) $\vec{r}\cdot\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$

(c) $\vec{r}\cdot\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$

Answer

(a) It is given that equation of the plane is

$$\vec{r}\cdot\left(\hat{i}+\hat{j}-\hat{k}\right)=2 \qquad \ldots(1)$$

For any arbitrary point P (*x*, *y*, *z*) on the plane, position vector $\vec{r}$ is given by,

$$\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$$

Substituting the value of $\vec{r}$ in equation (1), we obtain

$$\left(x\hat{i}+y\hat{j}-z\hat{k}\right)\cdot\left(\hat{i}+\hat{j}-\hat{k}\right)=2$$
$$\Rightarrow x+y-z=2$$

This is the Cartesian equation of the plane.

(b) $\vec{r}\cdot\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1 \quad \ldots(1)$

For any arbitrary point P (x, y, z) on the plane, position vector $\vec{r}$ is given by,

$$\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$$

Substituting the value of $\vec{r}$ in equation (1), we obtain

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(2\hat{i}+3\hat{j}-4\hat{k}\right)=1$$
$$\Rightarrow 2x+3y-4z=1$$

This is the Cartesian equation of the plane.

(c) $\vec{r}\cdot\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15 \quad \ldots(1)$

For any arbitrary point P (x, y, z) on the plane, position vector $\vec{r}$ is given by,

$$\vec{r}=x\hat{i}+y\hat{j}-z\hat{k}$$

Substituting the value of $\vec{r}$ in equation (1), we obtain

$$\left(x\hat{i}+y\hat{j}-z\hat{k}\right)\cdot\left[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}\right]=15$$
$$\Rightarrow (s-2t)x+(3-t)y+(2s+t)z=15$$

This is the Cartesian equation of the given plane.

Question 4:

In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

(a) $2x+3y+4z-12=0$ **(b)** $3y+4z-6=0$

(c) $x+y+z=1$ **(d)** $5y+8=0$

Answer

(a) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$2x + 3y + 4z - 12 = 0$

$\Rightarrow 2x + 3y + 4z = 12$ … (1)

The direction ratios of normal are 2, 3, and 4.

$$\therefore \sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{29}$$

Dividing both sides of equation (1) by $\sqrt{29}$, we obtain

$$\frac{2}{\sqrt{29}}x + \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{12}{\sqrt{29}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

(ld, md, nd).

Therefore, the coordinates of the foot of the perpendicular are

$$\left(\frac{2}{\sqrt{29}}.\frac{12}{\sqrt{29}}, \frac{3}{\sqrt{29}}.\frac{12}{\sqrt{29}}, \frac{4}{\sqrt{29}}.\frac{12}{\sqrt{29}}\right) \text{ i.e., } \left(\frac{24}{29}, \frac{36}{49}, \frac{48}{29}\right).$$

(b) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$3y + 4z - 6 = 0$

$\Rightarrow 0x + 3y + 4z = 6$ … (1)

The direction ratios of the normal are 0, 3, and 4.

$$\therefore \sqrt{0 + 3^2 + 4^2} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$0x + \frac{3}{5}y + \frac{4}{5}z = \frac{6}{5}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

(ld, md, nd).

Therefore, the coordinates of the foot of the perpendicular are

$$\left(0, \frac{3}{5}.\frac{6}{5}, \frac{4}{5}.\frac{6}{5}\right) \text{ i.e., } \left(0, \frac{18}{25}, \frac{24}{25}\right).$$

(c) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$x + y + z = 1$ … (1)

The direction ratios of the normal are 1, 1, and 1.

$$\therefore \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

(ld, md, nd).

Therefore, the coordinates of the foot of the perpendicular are

$$\left(\frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.\frac{1}{\sqrt{3}}\right) \text{ i.e., } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

(d) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1).

$5y + 8 = 0$

$\Rightarrow 0x - 5y + 0z = 8$ … (1)

The direction ratios of the normal are 0, −5, and 0.

$\therefore \sqrt{0+(-5)^2+0}=5$

Dividing both sides of equation (1) by 5, we obtain

$-y=\frac{8}{5}$

This equation is of the form *lx* + *my* + *nz* = *d*, where *l*, *m*, *n* are the direction cosines of normal to the plane and *d* is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

(*ld*, *md*, *nd*).

Therefore, the coordinates of the foot of the perpendicular are

$\left(0, -1\left(\frac{8}{5}\right), 0\right)$ i.e., $\left(0, -\frac{8}{5}, 0\right)$.

Question 5:

Find the vector and Cartesian equation of the planes

(a) that passes through the point (1, 0, −2) and the normal to the plane is $\hat{i}+\hat{j}-\hat{k}$.

(b) that passes through the point (1, 4, 6) and the normal vector to the plane is $\hat{i}-2\hat{j}+\hat{k}$.

Answer

(a) The position vector of point (1, 0, −2) is $\vec{a}=\hat{i}-2\hat{k}$

The normal vector $\vec{N}$ perpendicular to the plane is $\vec{N}=\hat{i}+\hat{j}-\hat{k}$

The vector equation of the plane is given by, $(\vec{r}-\vec{a}).\vec{N}=0$

$\Rightarrow \left[\vec{r}-\left(\hat{i}-2\hat{k}\right)\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0 \qquad ...(1)$

$\vec{r}$ is the position vector of any point P (*x*, *y*, *z*) in the plane.

$\therefore \vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

Therefore, equation (1) becomes

$$\left[\left(x\hat{i}+y\hat{j}+z\hat{k}\right)-\left(\hat{i}-2\hat{k}\right)\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0$$

$$\Rightarrow\left[(x-1)\hat{i}+y\hat{j}+(z+2)\hat{k}\right].\left(\hat{i}+\hat{j}-\hat{k}\right)=0$$

$$\Rightarrow(x-1)+y-(z+2)=0$$

$$\Rightarrow x+y-z-3=0$$

$$\Rightarrow x+y-z=3$$

This is the Cartesian equation of the required plane.

(b) The position vector of the point (1, 4, 6) is $\vec{a}=\hat{i}+4\hat{j}+6\hat{k}$

The normal vector $\vec{N}$ perpendicular to the plane is $\vec{N}=\hat{i}-2\hat{j}+\hat{k}$

The vector equation of the plane is given by, $(\vec{r}-\vec{a}).\vec{N}=0$

$$\Rightarrow\left[\vec{r}-\left(\hat{i}+4\hat{j}+6\hat{k}\right)\right].\left(\hat{i}-2\hat{j}+\hat{k}\right)=0 \qquad ...(1)$$

$\vec{r}$ is the position vector of any point P (*x*, *y*, *z*) in the plane.

$$\therefore\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$$

Therefore, equation (1) becomes

$$\left[\left(x\hat{i}+y\hat{j}+z\hat{k}\right)-\left(\hat{i}+4\hat{j}+6\hat{k}\right)\right].\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$$

$$\Rightarrow\left[(x-1)\hat{i}+(y-4)\hat{j}+(z-6)\hat{k}\right]\cdot\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$$

$$\Rightarrow(x-1)-2(y-4)+(z-6)=0$$

$$\Rightarrow x-2y+z+1=0$$

This is the Cartesian equation of the required plane.

Question 6:

Find the equations of the planes that passes through three points.

(a) (1, 1, −1), (6, 4, −5), (−4, −2, 3)

(b) (1, 1, 0), (1, 2, 1), (−2, 2, −1)

Answer

(a) The given points are A (1, 1, −1), B (6, 4, −5), and C (−4, −2, 3).

$$\begin{vmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{vmatrix} = (12-10)-(18-20)-(-12+16)$$
$$= 2+2-4$$
$$= 0$$

Since A, B, C are collinear points, there will be infinite number of planes passing through the given points.

(b) The given points are A (1, 1, 0), B (1, 2, 1), and C (−2, 2, −1).

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix} = (-2-2)-(2+2) = -8 \neq 0$$

Therefore, a plane will pass through the points A, B, and C.

It is known that the equation of the plane through the points, $(x_1, y_1, z_1), (x_2, y_2, z_2)$, and (x_3, y_3, z_3), is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (-2)(x-1)-3(y-1)+3z = 0$$
$$\Rightarrow -2x-3y+3z+2+3 = 0$$
$$\Rightarrow -2x-3y+3z = -5$$
$$\Rightarrow 2x+3y-3z = 5$$

This is the Cartesian equation of the required plane.

Question 7:

Find the intercepts cut off by the plane $2x+y-z=5$

Answer

$$2x+y-z=5 \qquad \ldots(1)$$

Dividing both sides of equation (1) by 5, we obtain

$$\frac{2}{5}x+\frac{y}{5}-\frac{z}{5}=1$$

$$\Rightarrow \frac{x}{\frac{5}{2}}+\frac{y}{5}+\frac{z}{-5}=1 \qquad ...(2)$$

It is known that the equation of a plane in intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, where *a*, *b*, *c* are the intercepts cut off by the plane at *x*, *y*, and *z* axes respectively.

Therefore, for the given equation,

$$a=\frac{5}{2}, b=5, \text{ and } c=-5$$

Thus, the intercepts cut off by the plane are $\frac{5}{2}$, 5, and -5.

Question 8:

Find the equation of the plane with intercept 3 on the *y*-axis and parallel to ZOX plane.

Answer

The equation of the plane ZOX is

y = 0

Any plane parallel to it is of the form, *y* = *a*

Since the *y*-intercept of the plane is 3,

∴ *a* = 3

Thus, the equation of the required plane is *y* = 3

Question 9:

Find the equation of the plane through the intersection of the planes $3x-y+2z-4=0$ and $x+y+z-2=0$ and the point (2, 2, 1)

Answer

The equation of any plane through the intersection of the planes,

$3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$, is

$(3x - y + 2z - 4) + \alpha(x + y + z - 2) = 0$, where $\alpha \in \mathrm{R}$ ...(1)

The plane passes through the point (2, 2, 1). Therefore, this point will satisfy equation (1).

$\therefore (3\times 2 - 2 + 2\times 1 - 4) + \alpha(2 + 2 + 1 - 2) = 0$

$\Rightarrow 2 + 3\alpha = 0$

$\Rightarrow \alpha = -\frac{2}{3}$

Substituting $\alpha = -\frac{2}{3}$ in equation (1), we obtain

$(3x - y + 2z - 4) - \frac{2}{3}(x + y + z - 2) = 0$

$\Rightarrow 3(3x - y + 2z - 4) - 2(x + y + z - 2) = 0$

$\Rightarrow (9x - 3y + 6z - 12) - 2(x + y + z - 2) = 0$

$\Rightarrow 7x - 5y + 4z - 8 = 0$

This is the required equation of the plane.

Question 10:

Find the vector equation of the plane passing through the intersection of the planes $\vec{r}\cdot(2\hat{i} + 2\hat{j} - 3\hat{k}) = 7,\ \vec{r}.(2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and through the point (2, 1, 3)

Answer

The equations of the planes are $\vec{r}\cdot(2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$ and $\vec{r}\cdot(2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$

$\Rightarrow \vec{r}\cdot(2\hat{i} + 2\hat{j} - 3\hat{k}) - 7 = 0$ ...(1)

$\vec{r}\cdot(2\hat{i} + 5\hat{j} + 3\hat{k}) - 9 = 0$ ...(2)

The equation of any plane through the intersection of the planes given in equations (1) and (2) is given by,

$\left[\vec{r}\cdot(2\hat{i} + 2\hat{j} - 3\hat{k}) - 7\right] + \lambda\left[\vec{r}\cdot(2\hat{i} + 5\hat{j} + 3\hat{k}) - 9\right] = 0$, where $\lambda \in R$

$\vec{r}\cdot\left[\left(2\hat{i}+2\hat{j}-3\hat{k}\right)+\lambda\left(2\hat{i}+5\hat{j}+3\hat{k}\right)\right]=9\lambda+7$

$\vec{r}\cdot\left[\left(2+2\lambda\right)\hat{i}+\left(2+5\lambda\right)\hat{j}+\left(3\lambda-3\right)\hat{k}\right]=9\lambda+7 \qquad ...(3)$

The plane passes through the point (2, 1, 3). Therefore, its position vector is given by,

$\vec{r}=2\hat{i}+2\hat{j}+3\hat{k}$

Substituting in equation (3), we obtain

$\left(2\hat{i}+\hat{j}-3\hat{k}\right).\left[\left(2+2\lambda\right)\hat{i}+\left(2+5\lambda\right)\hat{j}+\left(3\lambda-3\right)\hat{k}\right]=9\lambda+7$

$\Rightarrow\left(2+2\lambda\right)+\left(2+5\lambda\right)+\left(3\lambda-3\right)=9\lambda+7$

$\Rightarrow 18\lambda-3=9\lambda+7$

$\Rightarrow 9\lambda=10$

$\Rightarrow \lambda=\frac{10}{9}$

Substituting $\lambda=\frac{10}{9}$ in equation (3), we obtain

$\vec{r}\cdot\left(\frac{38}{9}\hat{i}+\frac{68}{9}\hat{j}+\frac{3}{9}\hat{k}\right)=17$

$\Rightarrow\vec{r}\cdot\left(38\hat{i}+68\hat{j}+3\hat{k}\right)=153$

This is the vector equation of the required plane.

Question 11:

Find the equation of the plane through the line of intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ which is perpendicular to the plane $x-y+z=0$

Answer

The equation of the plane through the intersection of the planes, $x+y+z=1$ and $2x+3y+4z=5$, is

$\left(x+y+z-1\right)+\lambda\left(2x+3y+4z-5\right)=0$

$\Rightarrow\left(2\lambda+1\right)x+\left(3\lambda+1\right)y+\left(4\lambda+1\right)z-\left(5\lambda+1\right)=0 \qquad ...(1)$

The direction ratios, a_1, b_1, c_1, of this plane are (2λ + 1), (3λ + 1), and (4λ + 1).

The plane in equation (1) is perpendicular to $x-y+z=0$

Its direction ratios, a_2, b_2, c_2, are 1, −1, and 1.

Since the planes are perpendicular,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$
$$\Rightarrow (2\lambda+1) - (3\lambda+1) + (4\lambda+1) = 0$$
$$\Rightarrow 3\lambda + 1 = 0$$
$$\Rightarrow \lambda = -\frac{1}{3}$$

Substituting $\lambda = -\frac{1}{3}$ in equation (1), we obtain

$$\frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} = 0$$
$$\Rightarrow x - z + 2 = 0$$

This is the required equation of the plane.

Question 12:

Find the angle between the planes whose vector equations are

$\vec{r}\cdot\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ and $\vec{r}\cdot\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$.

Answer

The equations of the given planes are $\vec{r}\cdot\left(2\hat{i}+2\hat{j}-3\hat{k}\right)=5$ and $\vec{r}\cdot\left(3\hat{i}-3\hat{j}+5\hat{k}\right)=3$

It is known that if $\vec{n}_1$ and $\vec{n}_2$ are normal to the planes, $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$, then the angle between them, Q, is given by,

$$\cos Q = \left|\frac{\vec{n}_1\cdot\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right| \qquad ...(1)$$

Here, $\vec{n}_1 = 2\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{n}_2 = 3\hat{i}-3\hat{j}+5\hat{k}$

$$\therefore \vec{n}_1\cdot\vec{n}_2 = \left(2\hat{i}+2\hat{j}-3\hat{k}\right)\left(3\hat{i}-3\hat{j}+5\hat{k}\right) = 2.3 + 2.(-3) + (-3).5 = -15$$

$$|\vec{n}_1| = \sqrt{(2)^2+(2)^2+(-3)^2} = \sqrt{17}$$

$$|\vec{n}_2| = \sqrt{(3)^2+(-3)^2+(5)^2} = \sqrt{43}$$

Substituting the value of $\vec{n} \cdot \vec{n}_2$, $|\vec{n}_1| \text{and} |\vec{n}_2|$ in equation (1), we obtain

$$\cos Q = \left| \frac{-15}{\sqrt{17} \cdot \sqrt{43}} \right|$$

$$\Rightarrow \cos Q = \frac{15}{\sqrt{731}}$$

$$\Rightarrow \cos Q^{-1} = \left(\frac{15}{\sqrt{731}} \right)$$

Question 13:

In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

(b) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

(c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

(d) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

(e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

Answer

The direction ratios of normal to the plane, $L_1 : a_1x + b_1y + c_1z = 0$, are a_1, b_1, c_1 and $L_2 : a_1x + b_2y + c_2z = 0$ are a_2, b_2, c_2.

$$L_1 \parallel L_2, \text{ if } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$L_1 \perp L_2, \text{ if } a_1a_2 + b_1b_2 + c_1c_2 = 0$$

The angle between L_1 and L_2 is given by,

$$Q = \cos^{-1} \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

(a) The equations of the planes are $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

Here, $a_1 = 7$, $b_1 = 5$, $c_1 = 6$

$a_2 = 3,\ b_2 = -1, c_2 = -10$

$a_1a_2 + b_1b_2 + c_1c_2 = 7\times3+5\times(-1)+6\times(-10) = -44 \neq 0$

Therefore, the given planes are not perpendicular.

$\frac{a_1}{a_2} = \frac{7}{3}, \frac{b_1}{b_2} = \frac{5}{-1} = -5, \frac{c_1}{c_2} = \frac{6}{-10} = \frac{-3}{5}$

It can be seen that, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq, \frac{c_1}{c_2}$

Therefore, the given planes are not parallel.

The angle between them is given by,

$$Q = \cos^{-1}\left|\frac{7\times3+5\times(-1)+6\times(-10)}{\sqrt{(7)^2+(5)^2+(6)^2}\times\sqrt{(3)^2+(-1)^2+(-10)^2}}\right|$$

$$= \cos^{-1}\left|\frac{21-5-60}{\sqrt{110}\times\sqrt{110}}\right|$$

$$= \cos^{-1}\frac{44}{110}$$

$$= \cos^{-1}\frac{2}{5}$$

(b) The equations of the planes are $2x+y+3z-2=0$ and $x-2y+5=0$

Here, $a_1 = 2,\ b_1 = 1,\ c_1 = 3$ and $a_2 = 1,\ b_2 = -2, c_2 = 0$

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 2\times1+1\times(-2)+3\times0 = 0$

Thus, the given planes are perpendicular to each other.

(c) The equations of the given planes are $2x-2y+4z+5=0$ and $3x-3y+6z-1=0$

Here, $a_1 = 2,\ b_1 - 2,\ c_1 = 4$ and

$a_2 = 3,\ b_2 = -3,\ c_2 = 6\quad a_1a_2 + b_1b_2 + c_1c_2 = 2\times3+(-2)(-3)+4\times6 = 6+6+24 = 36 \neq 0$

Thus, the given planes are not perpendicular to each other.

$\frac{a_1}{a_2} = \frac{2}{3}, \frac{b_1}{b_2} = \frac{-2}{-3} = \frac{2}{3} \text{and} \frac{c_1}{c_2} = \frac{4}{6} = \frac{2}{3}$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Thus, the given planes are parallel to each other.

(d) The equations of the planes are $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

Here, $a_1 = 2, b_1 = -1, c_1 = 3$ and $a_2 = 2, b_2 = -1, c_2 = 3$

$\frac{a_1}{a_2} = \frac{2}{2} = 1, \frac{b_1}{b_2} = \frac{-1}{-1} = 1 \text{ and } \frac{c_1}{c_2} = \frac{3}{3} = 1$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Thus, the given lines are parallel to each other.

(e) The equations of the given planes are $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

Here, $a_1 = 4, b_1 = 8, c_1 = 1$ and $a_2 = 0, b_2 = 1, c_2 = 1$

$a_1a_2 + b_1b_2 + c_1c_2 = 4 \times 0 + 8 \times 1 + 1 = 9 \neq 0$

Therefore, the given lines are not perpendicular to each other.

$\frac{a_1}{a_2} = \frac{4}{0}, \ \frac{b_1}{b_2} = \frac{8}{1} = 8, \ \frac{c_1}{c_2} = \frac{1}{1} = 1$

$\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq, \frac{c_1}{c_2}$

Therefore, the given lines are not parallel to each other.

The angle between the planes is given by,

$$Q = \cos^{-1}\left|\frac{4 \times 0 + 8 \times 1 + 1 \times 1}{\sqrt{4^2 + 8^2 + 1^2} \times \sqrt{0^2 + 1^2 + 1^2}}\right| = \cos^{-1}\left|\frac{9}{9 \times \sqrt{2}}\right| = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$$

Question 14:

In the following cases, find the distance of each of the given points from the corresponding given plane.

Point Plane

(a) (0, 0, 0) $3x-4y+12z=3$

(b) (3, −2, 1) $2x-y+2z+3=0$

(c) (2, 3, −5) $x+2y-2z=9$

(d) (−6, 0, 0) $2x-3y+6z-2=0$

Answer

It is known that the distance between a point, $p(x_1, y_1, z_1)$, and a plane, $Ax + By + Cz = D$, is given by,

$$d=\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right| \qquad ...(1)$$

(a) The given point is (0, 0, 0) and the plane is $3x-4y+12z=3$

$$\therefore d=\left|\frac{3\times0-4\times0+12\times0-3}{\sqrt{(3)^2+(-4)^2+(12)^2}}\right|=\frac{3}{\sqrt{169}}=\frac{3}{13}$$

(b) The given point is (3, − 2, 1) and the plane is $2x-y+2z+3=0$

$$\therefore d=\left|\frac{2\times3-(-2)+2\times1+3}{\sqrt{(2)^2+(-1)^2+(2)^2}}\right|=\left|\frac{13}{3}\right|=\frac{13}{3}$$

(c) The given point is (2, 3, −5) and the plane is $x+2y-2z=9$

$$\therefore d=\left|\frac{2+2\times3-2(-5)-9}{\sqrt{(1)^2+(2)^2+(-2)^2}}\right|=\frac{9}{3}=3$$

(d) The given point is (−6, 0, 0) and the plane is $2x-3y+6z-2=0$

$$d = \left| \frac{2(-6) - 3 \times 0 + 6 \times 0 - 2}{\sqrt{(2)^2 + (-3)^2 + (6)^2}} \right| = \left| \frac{-14}{\sqrt{49}} \right| = \frac{14}{7} = 2$$