

Exercise 11.2

Question 1:

Show that the three lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Answer

Two lines with direction cosines, l_1, m_1, n_1 and l_2, m_2, n_2 , are perpendicular to each other, if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(i) For the lines with direction cosines, $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$, we obtain

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{12}{13} \times \frac{4}{13} + \left(\frac{-3}{13}\right) \times \frac{12}{13} + \left(\frac{-4}{13}\right) \times \frac{3}{13} \\ &= \frac{48}{169} - \frac{36}{169} - \frac{12}{169} \\ &= 0 \end{aligned}$$

Therefore, the lines are perpendicular.

(ii) For the lines with direction cosines, $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$, we obtain

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{4}{13} \times \frac{3}{13} + \frac{12}{13} \times \left(\frac{-4}{13}\right) + \frac{3}{13} \times \frac{12}{13} \\ &= \frac{12}{169} - \frac{48}{169} + \frac{36}{169} \\ &= 0 \end{aligned}$$

Therefore, the lines are perpendicular.

(iii) For the lines with direction cosines, $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ and $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$, we obtain

$$\begin{aligned}
 l_1l_2 + m_1m_2 + n_1n_2 &= \left(\frac{3}{13}\right) \times \left(\frac{12}{13}\right) + \left(\frac{-4}{13}\right) \times \left(\frac{-3}{13}\right) + \left(\frac{12}{13}\right) \times \left(\frac{-4}{13}\right) \\
 &= \frac{36}{169} + \frac{12}{169} - \frac{48}{169} \\
 &= 0
 \end{aligned}$$

Therefore, the lines are perpendicular.

Thus, all the lines are mutually perpendicular.

Question 2:

Show that the line through the points $(1, -1, 2)$ $(3, 4, -2)$ is perpendicular to the line through the points $(0, 3, 2)$ and $(3, 5, 6)$.

Answer

Let AB be the line joining the points, $(1, -1, 2)$ and $(3, 4, -2)$, and CD be the line joining the points, $(0, 3, 2)$ and $(3, 5, 6)$.

The direction ratios, a_1, b_1, c_1 , of AB are $(3 - 1)$, $(4 - (-1))$, and $(-2 - 2)$ i.e., 2, 5, and -4.

The direction ratios, a_2, b_2, c_2 , of CD are $(3 - 0)$, $(5 - 3)$, and $(6 - 2)$ i.e., 3, 2, and 4.

AB and CD will be perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + 5 \times 2 + (-4) \times 4$$

$$= 6 + 10 - 16$$

$$= 0$$

Therefore, AB and CD are perpendicular to each other.

Question 3:

Show that the line through the points $(4, 7, 8)$ $(2, 3, 4)$ is parallel to the line through the points $(-1, -2, 1)$, $(1, 2, 5)$.

Answer

Let AB be the line through the points, $(4, 7, 8)$ and $(2, 3, 4)$, and CD be the line through the points, $(-1, -2, 1)$ and $(1, 2, 5)$.

The directions ratios, a_1, b_1, c_1 , of AB are $(2 - 4)$, $(3 - 7)$, and $(4 - 8)$ i.e., -2, -4, and -4.

The direction ratios, a_2, b_2, c_2 , of CD are $(1 - (-1))$, $(2 - (-2))$, and $(5 - 1)$ i.e., 2, 4, and 4.

AB will be parallel to CD, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$\frac{a_1}{a_2} = \frac{-2}{2} = -1$$

$$\frac{b_1}{b_2} = \frac{-4}{4} = -1$$

$$\frac{c_1}{c_2} = \frac{-4}{4} = -1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, AB is parallel to CD.

Question 4:

Find the equation of the line which passes through the point (1, 2, 3) and is parallel to

the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

Answer

It is given that the line passes through the point A (1, 2, 3). Therefore, the position

vector through A is $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$$

It is known that the line which passes through point A and parallel to \vec{b} is given by

$\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a constant.

$$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$$

This is the required equation of the line.

Question 5:

Find the equation of the line in vector and in Cartesian form that passes through the

point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.

Answer

It is given that the line passes through the point with position vector

$$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k} \quad \dots(1)$$

$$\vec{b} = \hat{i} + 2\hat{j} - \hat{k} \quad \dots(2)$$

It is known that a line through a point with position vector \vec{a} and parallel to \vec{b} is given by the equation, $\vec{r} = \vec{a} + \lambda\vec{b}$

$$\Rightarrow \vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + 2\hat{j} - \hat{k})$$

This is the required equation of the line in vector form.

$$\vec{r} = x\hat{i} - y\hat{j} + z\hat{k}$$

$$\Rightarrow x\hat{i} - y\hat{j} + z\hat{k} = (\lambda + 2)\hat{i} + (2\lambda - 1)\hat{j} + (-\lambda + 4)\hat{k}$$

Eliminating λ , we obtain the Cartesian form equation as

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the required equation of the given line in Cartesian form.

Question 6:

Find the Cartesian equation of the line which passes through the point

$$(-2, 4, -5) \text{ and parallel to the line given by } \frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

Answer

It is given that the line passes through the point $(-2, 4, -5)$ and is parallel to

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

The direction ratios of the line, $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$, are 3, 5, and 6.

The required line is parallel to $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Therefore, its direction ratios are $3k, 5k,$ and $6k,$ where $k \neq 0$

It is known that the equation of the line through the point (x_1, y_1, z_1) and with direction

ratios, $a, b, c,$ is given by $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Therefore the equation of the required line is

$$\frac{x+2}{3k} = \frac{y-4}{5k} = \frac{z+5}{6k}$$

$$\Rightarrow \frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6} = k$$

Question 7:

The Cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form.

Answer

The Cartesian equation of the line is

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2} \quad \dots (1)$$

The given line passes through the point $(5, -4, 6)$. The position vector of this point is

$$\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$$

Also, the direction ratios of the given line are 3, 7, and 2.

This means that the line is in the direction of vector, $\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

It is known that the line through position vector \vec{a} and in the direction of the vector \vec{b} is

given by the equation, $\vec{r} = \vec{a} + \lambda\vec{b}, \lambda \in R$

$$\Rightarrow \vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$

This is the required equation of the given line in vector form.

Question 8:

Find the vector and the Cartesian equations of the lines that pass through the origin and $(5, -2, 3)$.

Answer

The required line passes through the origin. Therefore, its position vector is given by,

$$\vec{a} = \vec{0} \quad \dots (1)$$

The direction ratios of the line through origin and $(5, -2, 3)$ are

$$(5 - 0) = 5, (-2 - 0) = -2, (3 - 0) = 3$$

The line is parallel to the vector given by the equation, $\vec{b} = 5\hat{i} - 2\hat{j} + 3\hat{k}$

The equation of the line in vector form through a point with position vector \vec{a} and parallel

to \vec{b} is, $\vec{r} = \vec{a} + \lambda\vec{b}$, $\lambda \in R$

$$\Rightarrow \vec{r} = \vec{0} + \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$$

$$\Rightarrow \vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$$

The equation of the line through the point (x_1, y_1, z_1) and direction ratios a, b, c is given

$$\text{by, } \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

Therefore, the equation of the required line in the Cartesian form is

$$\frac{x-0}{5} = \frac{y-0}{-2} = \frac{z-0}{3}$$

$$\Rightarrow \frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$$

Question 9:

Find the vector and the Cartesian equations of the line that passes through the points $(3, -2, -5), (3, -2, 6)$.

Answer

Let the line passing through the points, P $(3, -2, -5)$ and Q $(3, -2, 6)$, be PQ.

Since PQ passes through P $(3, -2, -5)$, its position vector is given by,

$$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$$

The direction ratios of PQ are given by,

$$(3 - 3) = 0, (-2 + 2) = 0, (6 + 5) = 11$$

The equation of the vector in the direction of PQ is

$$\vec{b} = 0\hat{i} - 0\hat{j} + 11\hat{k} = 11\hat{k}$$

The equation of PQ in vector form is given by, $\vec{r} = \vec{a} + \lambda\vec{b}$, $\lambda \in R$

$$\Rightarrow \vec{r} = (3\hat{i} - 2\hat{j} - 5\hat{k}) + 11\lambda\hat{k}$$

The equation of PQ in Cartesian form is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \text{ i.e.,}$$

$$\frac{x-3}{0} = \frac{y+2}{0} = \frac{z+5}{11}$$

Question 10:

Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$ and

$$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$ and

$$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$$

Answer

(i) Let Q be the angle between the given lines.

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

The angle between the given pairs of lines is given by,

The given lines are parallel to the vectors, $\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$, respectively.

$$\therefore |\vec{b}_1| = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$|\vec{b}_2| = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$$

$$\begin{aligned} \vec{b}_1 \cdot \vec{b}_2 &= (3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k}) \\ &= 3 \times 1 + 2 \times 2 + 6 \times 2 \\ &= 3 + 4 + 12 \\ &= 19 \end{aligned}$$

$$\Rightarrow \cos Q = \frac{19}{7 \times 3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{19}{21}\right)$$

(ii) The given lines are parallel to the vectors, $\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$, respectively.

$$\therefore |\vec{b}_1| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$|\vec{b}_2| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\begin{aligned} \vec{b}_1 \cdot \vec{b}_2 &= (\hat{i} - \hat{j} - 2\hat{k}) \cdot (3\hat{i} - 5\hat{j} - 4\hat{k}) \\ &= 1 \cdot 3 - 1(-5) - 2(-4) \\ &= 3 + 5 + 8 \\ &= 16 \end{aligned}$$

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

$$\Rightarrow \cos Q = \frac{16}{\sqrt{6} \cdot 5\sqrt{2}} = \frac{16}{\sqrt{2} \cdot \sqrt{3} \cdot 5\sqrt{2}} = \frac{16}{10\sqrt{3}}$$

$$\Rightarrow \cos Q = \frac{8}{5\sqrt{3}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

Question 11:

Find the angle between the following pairs of lines:

$$(i) \quad \frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$$

$$(ii) \quad \frac{x}{2} = \frac{y}{2} = \frac{z}{1} \text{ and } \frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$$

i. Answer

ii. Let \vec{b}_1 and \vec{b}_2 be the vectors parallel to the pair of lines,

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}, \text{ respectively.}$$

$$\therefore \vec{b}_1 = 2\hat{i} + 5\hat{j} - 3\hat{k} \text{ and } \vec{b}_2 = -\hat{i} + 8\hat{j} + 4\hat{k}$$

$$|\vec{b}_1| = \sqrt{(2)^2 + (5)^2 + (-3)^2} = \sqrt{38}$$

$$|\vec{b}_2| = \sqrt{(-1)^2 + (8)^2 + (4)^2} = \sqrt{81} = 9$$

$$\begin{aligned} \vec{b}_1 \cdot \vec{b}_2 &= (2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (-\hat{i} + 8\hat{j} + 4\hat{k}) \\ &= 2(-1) + 5 \times 8 + (-3) \cdot 4 \\ &= -2 + 40 - 12 \\ &= 26 \end{aligned}$$

The angle, Q , between the given pair of lines is given by the relation,

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

$$\Rightarrow \cos Q = \frac{26}{9\sqrt{38}}$$

$$\Rightarrow Q = \cos^{-1} \left(\frac{26}{9\sqrt{38}} \right)$$

(ii) Let \vec{b}_1, \vec{b}_2 be the vectors parallel to the given pair of lines, $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and

$$\frac{x-5}{4} = \frac{y-5}{1} = \frac{z-3}{8}, \text{ respectively.}$$

$$\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$$

$$\therefore |\vec{b}_1| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{9} = 3$$

$$|\vec{b}_2| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$$

$$\begin{aligned}\vec{b}_1 \cdot \vec{b}_2 &= (2\hat{i} + 2\hat{j} + \hat{k}) \cdot (4\hat{i} + \hat{j} + 8\hat{k}) \\ &= 2 \times 4 + 2 \times 1 + 1 \times 8 \\ &= 8 + 2 + 8 \\ &= 18\end{aligned}$$

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

If Q is the angle between the given pair of lines, then

$$\Rightarrow \cos Q = \frac{18}{3 \times 9} = \frac{2}{3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{2}{3}\right)$$

Question 12:

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$$

Find the values of p so the line

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

are at right angles.

Answer

The given equations can be written in the standard form as

$$\frac{x-1}{-3} = \frac{y-2}{\frac{2p}{7}} = \frac{z-3}{2} \quad \frac{x-1}{\frac{-3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$$

and

The direction ratios of the lines are $-3, \frac{2p}{7}, 2$ and $\frac{-3p}{7}, 1, -5$ respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 , are perpendicular to each other, if

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

$$\therefore (-3) \cdot \left(\frac{-3p}{7}\right) + \left(\frac{2p}{7}\right) \cdot (1) + 2 \cdot (-5) = 0$$

$$\Rightarrow \frac{9p}{7} + \frac{2p}{7} = 10$$

$$\Rightarrow 11p = 70$$

$$\Rightarrow p = \frac{70}{11}$$

Thus, the value of p is $\frac{70}{11}$.

Question 13:

Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

Answer

The equations of the given lines are $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

The direction ratios of the given lines are 7, -5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 , are perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$\therefore 7 \times 1 + (-5) \times 2 + 1 \times 3$$

$$= 7 - 10 + 3$$

$$= 0$$

Therefore, the given lines are perpendicular to each other.

Question 14:

Find the shortest distance between the lines

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and}$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

Answer

The equations of the given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \frac{\left| (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) \right|}{\left| \vec{b}_1 \times \vec{b}_2 \right|} \quad \dots(1)$$

Comparing the given equations, we obtain

$$\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b}_1 = \hat{i} - \hat{j} + \hat{k}$$

$$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$$

$$\vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 2\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\vec{b}_1 \times \vec{b}_2 = (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k} = -3\hat{i} + 3\hat{k}$$

$$\Rightarrow \left| \vec{b}_1 \times \vec{b}_2 \right| = \sqrt{(-3)^2 + (3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

Substituting all the values in equation (1), we obtain

$$d = \left| \frac{(-3\hat{i} + 3\hat{k}) \cdot (\hat{i} - 3\hat{j} - 2\hat{k})}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \left| \frac{-3 \cdot 1 + 3(-2)}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \left| \frac{-9}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

Question 15:

Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

Answer

The given lines are $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

It is known that the shortest distance between the two lines,

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$, is given by,

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \quad \dots(1)$$

Comparing the given equations, we obtain

$$x_1 = -1, y_1 = -1, z_1 = -1$$

$$a_1 = 7, b_1 = -6, c_1 = 1$$

$$x_2 = 3, y_2 = 5, z_2 = 7$$

$$a_2 = 1, b_2 = -2, c_2 = 1$$

$$\begin{aligned} \text{Then, } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} &= \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix} \\ &= 4(-6+2) - 6(7-1) + 8(-14+6) \\ &= -16 - 36 - 64 \\ &= -116 \end{aligned}$$

$$\begin{aligned} \Rightarrow \sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2} &= \sqrt{(-6+2)^2 + (1+7)^2 + (-14+6)^2} \\ &= \sqrt{16+36+64} \\ &= \sqrt{116} \\ &= 2\sqrt{29} \end{aligned}$$

Substituting all the values in equation (1), we obtain

$$d = \frac{-116}{2\sqrt{29}} = \frac{-58}{\sqrt{29}} = \frac{-2 \times 29}{\sqrt{29}} = -2\sqrt{29}$$

Since distance is always non-negative, the distance between the given lines is

$$2\sqrt{29} \text{ units.}$$

Question 16:

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$$

$$\text{and } \vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

Answer

$$\text{The given lines are } \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and } \vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \frac{\left| (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) \right|}{\left| \vec{b}_1 \times \vec{b}_2 \right|} \quad \dots(1)$$

Comparing the given equations with $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$, we obtain

$$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$$

$$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$$

$$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{i} - (1-4)\hat{j} + (3+6)\hat{k} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow \left| \vec{b}_1 \times \vec{b}_2 \right| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

$$\begin{aligned} (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) &= (-9\hat{i} + 3\hat{j} + 9\hat{k}) \cdot (3\hat{i} + 3\hat{j} + 3\hat{k}) \\ &= -9 \times 3 + 3 \times 3 + 9 \times 3 \\ &= 9 \end{aligned}$$

Substituting all the values in equation (1), we obtain

$$d = \frac{9}{3\sqrt{19}} = \frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

Question 17:

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \text{ and}$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

Answer

The given lines are

$$\begin{aligned}\vec{r} &= (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \\ \Rightarrow \vec{r} &= (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) \quad \dots(1)\end{aligned}$$

$$\begin{aligned}\vec{r} &= (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k} \\ \Rightarrow \vec{r} &= (\hat{i} - \hat{j} + \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) \quad \dots(2)\end{aligned}$$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \frac{\left| (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) \right|}{\left| \vec{b}_1 \times \vec{b}_2 \right|} \quad \dots(3)$$

For the given equations,

$$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$$

$$\vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{j} - 4\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = (-2+4)\hat{i} - (2+2)\hat{j} + (-2-1)\hat{k} = 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$\Rightarrow \left| \vec{b}_1 \times \vec{b}_2 \right| = \sqrt{(2)^2 + (-4)^2 + (-3)^2} = \sqrt{4+16+9} = \sqrt{29}$$

$$\therefore (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (2\hat{i} - 4\hat{j} - 3\hat{k}) \cdot (\hat{j} - 4\hat{k}) = -4 + 12 = 8$$

Substituting all the values in equation (3), we obtain

$$d = \frac{8}{\sqrt{29}} = \frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.