Exercise 13.1

Question 1:

Evaluate the Given limit: $\lim_{x\to 3} x+3$

Answer

$\lim_{x\to 3} x+3 = 3+3 = 6$

Question 2:

Evaluate the Given limit: $\lim_{x\to \pi}\left(x-\frac{22}{7}\right)$

Answer

$\lim_{x\to \pi}\left(x-\frac{22}{7}\right) = \left(\pi-\frac{22}{7}\right)$

Question 3:

Evaluate the Given limit: $\lim_{r\to 1}\pi r^2$

Answer

$\lim_{r\to 1}\pi r^2 = \pi (1)^2 = \pi$

Question 4:

Evaluate the Given limit: $\lim_{x\to 4}\frac{4x+3}{x-2}$

Answer

$\lim_{x\to 4}\frac{4x+3}{x-2} = \frac{4(4)+3}{4-2} = \frac{16+3}{2} = \frac{19}{2}$

Question 5:

Evaluate the Given limit: $\lim_{x\to -1}\frac{x^{10}+x^5+1}{x-1}$

Answer

$$\lim_{x\to -1}\frac{x^{10}+x^{5}+1}{x-1}=\frac{(-1)^{10}+(-1)^{5}+1}{-1-1}=\frac{1-1+1}{-2}=-\frac{1}{2}$$

Question 6:

Evaluate the Given limit: $\lim_{x\to 0}\frac{(x+1)^{5}-1}{x}$

Answer

$$\lim_{x\to 0}\frac{(x+1)^{5}-1}{x}$$

Put $x + 1 = y$ so that $y \to 1$ as $x \to 0$.

$$\text{Accordingly, } \lim_{x\to 0}\frac{(x+1)^{5}-1}{x}=\lim_{y\to 1}\frac{y^{5}-1}{y-1}$$

$$=\lim_{y\to 1}\frac{y^{5}-1^{5}}{y-1}$$

$$=5.1^{5-1} \qquad \left[\lim_{x\to a}\frac{x^{n}-a^{n}}{x-a}=na^{n-1}\right]$$

$$=5$$

$$\therefore \lim_{x\to 0}\frac{(x+5)^{5}-1}{x}=5$$

Question 7:

Evaluate the Given limit: $\lim_{x\to 2}\frac{3x^{2}-x-10}{x^{2}-4}$

Answer

At $x = 2$, the value of the given rational function takes the form $\frac{0}{0}$.

$$\therefore \lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$$

$$=\lim_{x\to 2}\frac{3x+5}{x+2}$$

$$=\frac{3(2)+5}{2+2}$$

$$=\frac{11}{4}$$

Question 8:

Evaluate the Given limit: $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Answer

At x = 2, the value of the given rational function takes the form $\frac{0}{0}$.

$$\therefore \lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$$

$$=\lim_{x\to 3}\frac{(x+3)(x^2+9)}{2x+1}$$

$$=\frac{(3+3)(3^2+9)}{2(3)+1}$$

$$=\frac{6\times 18}{7}$$

$$=\frac{108}{7}$$

Question 9:

Evaluate the Given limit: $\lim_{x\to 0}\frac{ax+b}{cx+1}$

Answer

$$\lim_{x\to 0}\frac{ax+b}{cx+1}=\frac{a(0)+b}{c(0)+1}=b$$

Question 10:

Evaluate the Given limit: $\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Answer

$$\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$$

At z = 1, the value of the given function takes the form $\frac{0}{0}$.

Put $z^{\frac{1}{6}}=x$ so that $z \to 1$ as $x \to 1$.

$$\text{Accordingly, } \lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}=\lim_{x\to 1}\frac{x^2-1}{x-1}$$

$$=\lim_{x\to 1}\frac{x^2-1^2}{x-1}$$

$$=2.1^{2-1} \qquad \left[\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=2$$

$$\therefore \lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}=2$$

Question 11:

Evaluate the Given limit: $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c\neq 0$

Answer

$$\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}=\frac{a(1)^2+b(1)+c}{c(1)^2+b(1)+a}$$
$$=\frac{a+b+c}{a+b+c}$$
$$=1 \qquad [a+b+c\neq 0]$$

Question 12:

Evaluate the Given limit: $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

Answer

$$\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$$

At x = –2, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}=\lim_{x\to -2}\frac{\left(\frac{2+x}{2x}\right)}{x+2}$$
$$=\lim_{x\to -2}\frac{1}{2x}$$
$$=\frac{1}{2(-2)}=\frac{-1}{4}$$

Question 13:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{bx}$

Answer

$$\lim_{x\to 0}\frac{\sin ax}{bx}$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x\to 0}\frac{\sin ax}{bx} = \lim_{x\to 0}\frac{\sin ax}{ax}\times\frac{ax}{bx}$$
$$= \lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)\times\left(\frac{a}{b}\right)$$
$$= \frac{a}{b}\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right) \qquad [x\to 0 \Rightarrow ax\to 0]$$
$$= \frac{a}{b}\times 1 \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$
$$= \frac{a}{b}$$

Question 14:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{\sin bx},\ a,\ b\neq 0$

Answer

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx},\ a,\ b\neq 0$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)\times ax}{\left(\frac{\sin bx}{bx}\right)\times bx}$$
$$= \left(\frac{a}{b}\right)\times\frac{\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim_{bx\to 0}\left(\frac{\sin bx}{bx}\right)} \qquad \left[\begin{array}{l} x\to 0\Rightarrow ax\to 0 \\ \text{and } x\to 0\Rightarrow bx\to 0\end{array}\right]$$
$$= \left(\frac{a}{b}\right)\times\frac{1}{1} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$
$$= \frac{a}{b}$$

Question 15:

Evaluate the Given limit: $\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$

Answer

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

It is seen that $x \to \pi \Rightarrow (\pi - x) \to 0$

$$\therefore \lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)} = \frac{1}{\pi}\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{(\pi-x)}$$

$$= \frac{1}{\pi}\times 1 \qquad \left[\lim_{y\to 0}\frac{\sin y}{y} = 1\right]$$

$$= \frac{1}{\pi}$$

Question 16:

Evaluate the given limit: $\lim_{x\to 0}\frac{\cos x}{\pi - x}$

Answer

$$\lim_{x\to 0}\frac{\cos x}{\pi - x} = \frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$$

Question 17:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\cos 2x - 1}{\cos x - 1}$

Answer

$$\lim_{x\to 0}\frac{\cos 2x - 1}{\cos x - 1}$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1} \qquad \left[\cos x=1-2\sin^2\frac{x}{2}\right]$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\left(\frac{\sin^2 x}{x^2}\right)\times x^2}{\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\times\frac{x^2}{4}}$$

$$=4\frac{\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)}{\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

$$=4\frac{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2}{\left(\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2} \qquad \left[x\to 0\Rightarrow\frac{x}{2}\to 0\right]$$

$$=4\frac{1^2}{1^2} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=4$$

Question 18:

Evaluate the Given limit: $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

Answer

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}=\frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}$$

$$=\frac{1}{b}\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{x\to 0}(a+\cos x)$$

$$=\frac{1}{b}\times\frac{1}{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)}\times\lim_{x\to 0}(a+\cos x)$$

$$=\frac{1}{b}\times(a+\cos 0) \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\frac{a+1}{b}$$

Question 19:

Evaluate the Given limit: $\lim_{x\to 0} x\sec x$

Answer

$$\lim_{x\to 0} x\sec x=\lim_{x\to 0}\frac{x}{\cos x}=\frac{0}{\cos 0}=\frac{0}{1}=0$$

Question 20:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}\ \ a,b,a+b\neq 0$

Answer

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}$$

$$=\lim_{x\to 0}\frac{\left(\dfrac{\sin ax}{ax}\right)ax+bx}{ax+bx\left(\dfrac{\sin bx}{bx}\right)}$$

$$=\frac{\left(\lim\limits_{ax\to 0}\dfrac{\sin ax}{ax}\right)\times\lim\limits_{x\to 0}(ax)+\lim\limits_{x\to 0}bx}{\lim\limits_{x\to 0}ax+\lim\limits_{x\to 0}bx\left(\lim\limits_{bx\to 0}\dfrac{\sin bx}{bx}\right)} \qquad [\text{As } x\to 0\Rightarrow ax\to 0 \text{ and } bx\to 0]$$

$$=\frac{\lim\limits_{x\to 0}(ax)+\lim\limits_{x\to 0}bx}{\lim\limits_{x\to 0}ax+\lim\limits_{x\to 0}bx} \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\frac{\lim\limits_{x\to 0}(ax+bx)}{\lim\limits_{x\to 0}(ax+bx)}$$

$$=\lim_{x\to 0}(1)$$

$$=1$$

Question 21:

Evaluate the Given limit: $\lim\limits_{x\to 0}(\operatorname{cosec} x-\cot x)$

Answer

At x = 0, the value of the given function takes the form $\infty-\infty$.

Now,

$$\lim_{x\to 0}(\operatorname{cosec} x - \cot x)$$
$$= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$
$$= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)$$
$$= \lim_{x\to 0}\frac{\left(\frac{1-\cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$
$$= \frac{\lim_{x\to 0}\frac{1-\cos x}{x}}{\lim_{x\to 0}\frac{\sin x}{x}}$$
$$= \frac{0}{1} \qquad \left[\lim_{x\to 0}\frac{1-\cos x}{x} = 0 \text{ and } \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$
$$= 0$$

Question 22:

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

Answer

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

At $x = \frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$.

Now, put $x - \frac{\pi}{2} = y$ so that $x \to \frac{\pi}{2},\ y \to 0$.

$$\therefore \lim_{x\to\frac{\pi}{2}} \frac{\tan 2x}{x-\frac{\pi}{2}} = \lim_{y\to 0} \frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$$

$$= \lim_{y\to 0} \frac{\tan(\pi+2y)}{y}$$

$$= \lim_{y\to 0} \frac{\tan 2y}{y} \qquad [\tan(\pi+2y) = \tan 2y]$$

$$= \lim_{y\to 0} \frac{\sin 2y}{y\cos 2y}$$

$$= \lim_{y\to 0} \left(\frac{\sin 2y}{2y} \times \frac{2}{\cos 2y} \right)$$

$$= \left(\lim_{2y\to 0} \frac{\sin 2y}{2y} \right) \times \lim_{y\to 0} \left(\frac{2}{\cos 2y} \right) \qquad [y \to 0 \Rightarrow 2y \to 0]$$

$$= 1 \times \frac{2}{\cos 0} \qquad \left[\lim_{x\to 0} \frac{\sin x}{x} = 1 \right]$$

$$= 1 \times \frac{2}{1}$$

$$= 2$$

Question 23:

Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$

Answer

The given function is

$f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}[2x+3] = 2(0)+3 = 3$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3(x+1) = 3(0+1) = 3$$

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = 3$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} 3(x+1) = 3(1+1) = 6$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1} 3(x+1) = 3(1+1) = 6$$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 6$$

Question 24:

Find $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$

Answer

The given function is

$$f(x) = \begin{cases} x^2 - 1, \ x \le 1 \\ -x^2 - 1, \ x > 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}\left[x^2 - 1\right] = 1^2 - 1 = 1 - 1 = 0$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}\left[-x^2 - 1\right] = -1^2 - 1 = -1 - 1 = -2$$

It is observed that $\lim_{x\to 1^-} f(x) \ne \lim_{x\to 1^+} f(x)$.

Hence, $\lim_{x\to 1} f(x)$ does not exist.

Question 25:

Evaluate $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

Answer

The given function is

$f(x) = \begin{cases} \frac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left[\frac{|x|}{x}\right]$$
$$= \lim_{x\to 0}\left(\frac{-x}{x}\right) \qquad \left[\text{When } x \text{ is negaitve, } |x| = -x\right]$$
$$= \lim_{x\to 0}(-1)$$
$$= -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left[\frac{|x|}{x}\right]$$
$$= \lim_{x\to 0}\left[\frac{x}{x}\right] \qquad \left[\text{When } x \text{ is positive, } |x| = x\right]$$
$$= \lim_{x\to 0}(1)$$
$$= 1$$

It is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Question 26:

Find $\lim_{x\to 0}$ $f(x)$, where $f(x)$ = $\begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Answer

The given function is

$$f(x)=\begin{cases}\frac{x}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\left[\frac{x}{|x|}\right]$$

$$=\lim_{x\to 0}\left[\frac{x}{-x}\right] \qquad [\text{When } x<0,\ |x|=-x]$$

$$=\lim_{x\to 0}(-1)$$

$$=-1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\left[\frac{x}{|x|}\right]$$

$$=\lim_{x\to 0}\left[\frac{x}{x}\right] \qquad [\text{When } x>0,\ |x|=x]$$

$$=\lim_{x\to 0}(1)$$

$$=1$$

It is observed that $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Question 27:

Find $\lim_{x\to 5}$ $f(x)$, where $f(x)$ = $|x|-5$

Answer

The given function is $f(x)$ = $|x|-5$.

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} \left[|x| - 5 \right]$$
$$= \lim_{x \to 5} (x - 5) \qquad \left[\text{When } x > 0,\ |x| = x \right]$$
$$= 5 - 5$$
$$= 0$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} \left(|x| - 5 \right)$$
$$= \lim_{x \to 5} (x - 5) \qquad \left[\text{When } x > 0,\ |x| = x \right]$$
$$= 5 - 5$$
$$= 0$$

$$\therefore \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = 0$$

Hence, $\lim_{x \to 5} f(x) = 0$

Question 28:

Suppose $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax & x > 1 \end{cases}$ and if $\lim_{x \to 1} f(x) = f(1)$ what are possible values of *a* and *b*?

Answer

The given function is

$$f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax & x > 1 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (a + bx) = a + b$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (b - ax) = b - a$$
$$f(1) = 4$$

It is given that $\lim_{x \to 1} f(x) = f(1)$.

$$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = f(1)$$

$\Rightarrow a + b = 4$ and $b - a = 4$

On solving these two equations, we obtain $a = 0$ and $b = 4$.

Thus, the respective possible values of *a* and *b* are 0 and 4.

Question 29:

Let $a_1,\ a_2,\ \ldots,\ a_n$ be fixed real numbers and define a function

$$f(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$$

What is $\lim_{x\to a_1} f(x)$? For some $a\neq a_1,\ a_2\ldots,\ a_n$, compute $\lim_{x\to a} f(x)$.

Answer

The given function is $f(x)=(x-a_1)(x-a_2)\ldots(x-a_n)$.

$$\begin{aligned}\lim_{x\to a_1} f(x)&=\lim_{x\to a_1}\left[(x-a_1)(x-a_2)\ldots(x-a_n)\right]\\&=\left[\lim_{x\to a_1}(x-a_1)\right]\left[\lim_{x\to a_1}(x-a_2)\right]\ldots\left[\lim_{x\to a_1}(x-a_n)\right]\\&=(a_1-a_1)(a_1-a_2)\ldots(a_1-a_n)=0\end{aligned}$$

$$\therefore \lim_{x\to a_1} f(x)=0$$

$$\begin{aligned}\text{Now, }\lim_{x\to a} f(x)&=\lim_{x\to a}\left[(x-a_1)(x-a_2)\ldots(x-a_n)\right]\\&=\left[\lim_{x\to a}(x-a_1)\right]\left[\lim_{x\to a}(x-a_2)\right]\ldots\left[\lim_{x\to a}(x-a_n)\right]\\&=(a-a_1)(a-a_2)\ldots..(a-a_n)\end{aligned}$$

$$\therefore \lim_{x\to a} f(x)=(a-a_1)(a-a_2)\ldots(a-a_n)$$

Question 30:

If $f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>0\end{cases}$.

For what value (s) of a does $\lim_{x\to a} f(x)$ exists?

Answer

The given function is

$$f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>0\end{cases}$$

When $a=0$,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(|x|+1)$$
$$=\lim_{x\to 0}(-x+1) \qquad [\text{If } x<0,\ |x|=-x]$$
$$=-0+1$$
$$=1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(|x|-1)$$
$$=\lim_{x\to 0}(x-1) \qquad [\text{If } x>0,\ |x|=x]$$
$$=0-1$$
$$=-1$$

Here, it is observed that $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)$.

$\therefore \lim_{x\to 0} f(x)$ does not exist.

When a < 0,

$$\lim_{x\to a^-} f(x)=\lim_{x\to a^-}(|x|+1)$$
$$=\lim_{x\to a}(-x+1) \qquad [x<a<0\Rightarrow\ |x|=-x]$$
$$=-a+1$$

$$\lim_{x\to a^+} f(x)=\lim_{x\to a^+}(|x|+1)$$
$$=\lim_{x\to a}(-x+1) \qquad [a<x<0\Rightarrow\ |x|=-x]$$
$$=-a+1$$

$\therefore \lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=-a+1$

Thus, limit of $f(x)$ exists at $x=a$, where $a<0$.

When a > 0

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} (|x| - 1)$$
$$= \lim_{x \to a} (x - 1) \qquad [0 < x < a \Rightarrow |x| = x]$$
$$= a - 1$$
$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} (|x| - 1)$$
$$= \lim_{x \to a} (x - 1) \qquad [0 < a < x \Rightarrow |x| = x]$$
$$= a - 1$$
$$\therefore \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = a - 1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a > 0$.

Thus, $\lim_{x \to a} f(x)$ exists for all $a \neq 0$.

Question 31:

If the function $f(x)$ satisfies $\lim_{x \to 1} \frac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \to 1} f(x)$.

Answer

$$\lim_{x \to 1} \frac{f(x) - 2}{x^2 - 1} = \pi$$
$$\Rightarrow \frac{\lim_{x \to 1} (f(x) - 2)}{\lim_{x \to 1} (x^2 - 1)} = \pi$$
$$\Rightarrow \lim_{x \to 1} (f(x) - 2) = \pi \lim_{x \to 1} (x^2 - 1)$$
$$\Rightarrow \lim_{x \to 1} (f(x) - 2) = \pi (1^2 - 1)$$
$$\Rightarrow \lim_{x \to 1} (f(x) - 2) = 0$$
$$\Rightarrow \lim_{x \to 1} f(x) - \lim_{x \to 1} 2 = 0$$
$$\Rightarrow \lim_{x \to 1} f(x) - 2 = 0$$
$$\therefore \lim_{x \to 1} f(x) = 2$$

Question 32:

If $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$. For what integers *m* and *n* does $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist?

Answer

The given function is

$$f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}(mx^2+n)$$
$$=m(0)^2+n$$
$$=n$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0}(nx+m)$$
$$=n(0)+m$$
$$=m.$$

Thus, $\lim_{x\to 0} f(x)$ exists if *m* = *n*.

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1}(nx+m)$$
$$=n(1)+m$$
$$=m+n$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}(nx^3+m)$$
$$=n(1)^3+m$$
$$=m+n$$

$$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x).$$

Thus, $\lim_{x\to 1} f(x)$ exists for any integral value of *m* and *n*.