Exercise 14.5

Question 1:

Show that the statement

p: "If x is a real number such that $x^3 + 4x = 0$, then x is 0" is true by

(i) direct method

(ii) method of contradiction

(iii) method of contrapositive

Answer

p: "If x is a real number such that $x^3 + 4x = 0$, then x is 0".

Let *q*: x is a real number such that $x^3 + 4x = 0$

r: x is 0.

(i) To show that statement *p* is true, we assume that *q* is true and then show that *r* is true.

Therefore, let statement *q* be true.

$\therefore x^3 + 4x = 0$

$x(x^2 + 4) = 0$

$\Rightarrow x = 0$ or $x^2 + 4 = 0$

However, since x is real, it is 0.

Thus, statement *r* is true.

Therefore, the given statement is true.

(ii) To show statement *p* to be true by contradiction, we assume that *p* is not true.

Let x be a real number such that $x^3 + 4x = 0$ and let x is not 0.

Therefore, $x^3 + 4x = 0$

$x(x^2 + 4) = 0$

$x = 0$ or $x^2 + 4 = 0$

$x = 0$ or $x^2 = -4$

However, x is real. Therefore, $x = 0$, which is a contradiction since we have assumed that x is not 0.

Thus, the given statement *p* is true.

(iii) To prove statement *p* to be true by contrapositive method, we assume that *r* is false and prove that *q* must be false.

Here, *r* is false implies that it is required to consider the negation of statement *r*. This obtains the following statement.

$\sim r$: x is not 0.

It can be seen that $(x^2 + 4)$ will always be positive.

$x \neq 0$ implies that the product of any positive real number with x is not zero.

Let us consider the product of x with $(x^2 + 4)$.

$\therefore x(x^2 + 4) \neq 0$

$\Rightarrow x^3 + 4x \neq 0$

This shows that statement q is not true.

Thus, it has been proved that

$\sim r \Rightarrow \sim q$

Therefore, the given statement p is true.

Question 2:

Show that the statement "For any real numbers a and b, $a^2 = b^2$ implies that $a = b$" is not true by giving a counter-example.

Answer

The given statement can be written in the form of "if-then" as follows.

If a and b are real numbers such that $a^2 = b^2$, then $a = b$.

Let p: a and b are real numbers such that $a^2 = b^2$.

q: $a = b$

The given statement has to be proved false. For this purpose, it has to be proved that if p, then $\sim q$. To show this, two real numbers, a and b, with $a^2 = b^2$ are required such that $a \neq b$.

Let $a = 1$ and $b = -1$

$a^2 = (1)^2 = 1$ and $b^2 = (-1)^2 = 1$

$\therefore a^2 = b^2$

However, $a \neq b$

Thus, it can be concluded that the given statement is false.

Question 3:

Show that the following statement is true by the method of contrapositive.

p: *If x is an integer and x^2 is even, then x is also even.*

Answer

p: If x is an integer and x^2 is even, then x is also even.

Let q: x is an integer and x^2 is even.

r: x is even.

To prove that p is true by contrapositive method, we assume that r is false, and prove that q is also false.

Let x is not even.

To prove that q is false, it has to be proved that x is not an integer or x^2 is not even.

x is not even implies that x^2 is also not even.

Therefore, statement q is false.

Thus, the given statement p is true.

Question 4:

By giving a counter example, show that the following statements are not true.

(i) p: If all the angles of a triangle are equal, then the triangle is an obtuse angled triangle.

(ii) q: The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2.

Answer

(i) The given statement is of the form "if q then r".

q: All the angles of a triangle are equal.

r: The triangle is an obtuse-angled triangle.

The given statement p has to be proved false. For this purpose, it has to be proved that if q, then $\sim r$.

To show this, angles of a triangle are required such that none of them is an obtuse angle.

It is known that the sum of all angles of a triangle is 180°. Therefore, if all the three angles are equal, then each of them is of measure 60°, which is not an obtuse angle.

In an equilateral triangle, the measure of all angles is equal. However, the triangle is not an obtuse-angled triangle.

Thus, it can be concluded that the given statement p is false.

(ii) The given statement is as follows.

q: The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2.

This statement has to be proved false. To show this, a counter example is required.

Consider $x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

One root of the equation $x^2 - 1 = 0$, i.e. the root $x = 1$, lies between 0 and 2.

Thus, the given statement is false.

Question 5:

Which of the following statements are true and which are false? In each case give a valid reason for saying so.

(i) *p*: Each radius of a circle is a chord of the circle.

(ii) *q*: The centre of a circle bisects each chord of the circle.

(iii) *r*: Circle is a particular case of an ellipse.

(iv) *s*: If *x* and *y* are integers such that $x > y$, then $-x < -y$.

(v) *t*: $\sqrt{11}$ is a rational number.

Answer

(i) The given statement *p* is false.

According to the definition of chord, it should intersect the circle at two distinct points.

(ii) The given statement *q* is false.

If the chord is not the diameter of the circle, then the centre will not bisect that chord.

In other words, the centre of a circle only bisects the diameter, which is the chord of the circle.

(iii) The equation of an ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we put $a = b = 1$, then we obtain

$x^2 + y^2 = 1$, which is an equation of a circle

Therefore, circle is a particular case of an ellipse.

Thus, statement *r* is true.

(iv) $x > y$

$\Rightarrow -x < -y$ (By a rule of inequality)

Thus, the given statement *s* is true.

(v) 11 is a prime number and we know that the square root of any prime number is an irrational number. Therefore, $\sqrt{11}$ is an irrational number.

Thus, the given statement *t* is false.