Miscellaneous Questions:

Question 1:

Let f: $\mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 10x + 7$. Find the function g: $\mathbf{R} \to \mathbf{R}$ such that $g \text{ o } f = f \text{ o } g = 1_{\mathbf{R}}$.

Answer

It is given that f: $\mathbf{R} \to \mathbf{R}$ is defined as $f(x) = 10x + 7$.

One-one:

Let $f(x) = f(y)$, where $x, y \in \mathbf{R}$.

$\Rightarrow 10x + 7 = 10y + 7$

$\Rightarrow x = y$

$\therefore$ f is a one-one function.

Onto:

For $y \in \mathbf{R}$, let $y = 10x + 7$.

$$\Rightarrow x = \frac{y-7}{10} \in \mathbf{R}$$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \frac{y-7}{10} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y.$$

$\therefore$ f is onto.

Therefore, f is one-one and onto.

Thus, f is an invertible function.

Let us define g: $\mathbf{R} \to \mathbf{R}$ as $g(y) = \frac{y-7}{10}$.

Now, we have:

$$gof(x) = g(f(x)) = g(10x+7) = \frac{(10x+7)-7}{10} = \frac{10x}{10} = 10$$

And,

$$fog(y) = f(g(y)) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right) + 7 = y - 7 + 7 = y$$

$$\therefore gof = \mathrm{I}_{\mathbf{R}} \text{ and } fog = \mathrm{I}_{\mathbf{R}}$$

Hence, the required function g: **R** → **R** is defined as $g(y)=\frac{y-7}{10}$.

Question 2:

Let f: W → W be defined as $f(n) = n - 1$, if is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

Answer

It is given that:

f: W → W is defined as $f(n)=\begin{cases} n-1, \text{ if } n \text{ is odd} \\ n+1, \text{ if } n \text{ is even} \end{cases}$

One-one:

Let $f(n) = f(m)$.

It can be observed that if n is odd and m is even, then we will have $n - 1 = m + 1$.

⇒ $n - m = 2$

However, this is impossible.

Similarly, the possibility of n being even and m being odd can also be ignored under a similar argument.

∴Both n and m must be either odd or even.

Now, if both n and m are odd, then we have:

$f(n) = f(m) \Rightarrow n - 1 = m - 1 \Rightarrow n = m$

Again, if both n and m are even, then we have:

$f(n) = f(m) \Rightarrow n + 1 = m + 1 \Rightarrow n = m$

∴f is one-one.

It is clear that any odd number $2r + 1$ in co-domain **N** is the image of $2r$ in domain **N** and any even number $2r$ in co-domain **N** is the image of $2r + 1$ in domain **N**.

∴f is onto.

Hence, f is an invertible function.

Let us define g: W → W as:

$$g(m)=\begin{cases} m+1, \text{ if } m \text{ is even} \\ m-1, \text{ if } m \text{ is odd} \end{cases}$$

Now, when n is odd:

$$gof(n) = g(f(n)) = g(n-1) = n-1+1 = n$$

And, when n is even:

$$gof(n) = g(f(n)) = g(n+1) = n+1-1 = n$$

Similarly, when m is odd:

$$fog(m) = f(g(m)) = f(m-1) = m-1+1 = m$$

When m is even:

$$fog(m) = f(g(m)) = f(m+1) = m+1-1 = m$$

$$\therefore \therefore gof = I_W \text{ and } fog = I_W$$

Thus, f is invertible and the inverse of f is given by $f^{-1} = g$, which is the same as f.

Hence, the inverse of f is f itself.

Question 3:

If f: **R** → **R** is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Answer

It is given that f: **R** → **R** is defined as $f(x) = x^2 - 3x + 2$.

$$\begin{aligned} f(f(x)) &= f(x^2-3x+2) \\ &= (x^2-3x+2)^2 - 3(x^2-3x+2)+2 \\ &= x^4+9x^2+4-6x^3-12x+4x^2-3x^2+9x-6+2 \\ &= x^4-6x^3+10x^2-3x \end{aligned}$$

Question 4:

Show that function f: **R** → $\{x \in$ **R**$: -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in$ **R** is one-one and onto function.

Answer

It is given that f: **R** → $\{x \in$ **R**$: -1 < x < 1\}$ is defined as $f(x) = \frac{x}{1+|x|}$, $x \in$ **R**.

Suppose $f(x) = f(y)$, where $x, y \in$ **R**.

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

It can be observed that if *x* is positive and *y* is negative, then we have:

$$\frac{x}{1+x} = \frac{y}{1-y} \Rightarrow 2xy = x - y$$

Since *x* is positive and *y* is negative:

$x > y \Rightarrow x - y > 0$

But, $2xy$ is negative.

Then, $2xy \neq x - y$.

Thus, the case of *x* being positive and *y* being negative can be ruled out.

Under a similar argument, *x* being negative and *y* being positive can also be ruled out

$\therefore$ *x* and *y* have to be either positive or negative.

When *x* and *y* are both positive, we have:

$$f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$$

When *x* and *y* are both negative, we have:

$$f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x - xy = y - yx \Rightarrow x = y$$

$\therefore$ *f* is one-one.

Now, let $y \in \mathbf{R}$ such that $-1 < y < 1$.

If *y* is negative, then there exists $x = \frac{y}{1+y} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y.$$

If *y* is positive, then there exists $x = \frac{y}{1-y} \in \mathbf{R}$ such that

$$f(x)=f\left(\frac{y}{1-y}\right)=\frac{\left(\frac{y}{1-y}\right)}{1+\left|\left(\frac{y}{1-y}\right)\right|}=\frac{\frac{y}{1-y}}{1+\frac{y}{1-y}}=\frac{y}{1-y+y}=y.$$

∴ *f* is onto.

Hence, *f* is one-one and onto.

Question 5:

Show that the function f: **R** → **R** given by $f(x) = x^3$ is injective.

Answer

f: **R** → **R** is given as $f(x) = x^3$.

Suppose $f(x) = f(y)$, where $x, y \in$ **R**.

$\Rightarrow x^3 = y^3$ … (1)

Now, we need to show that $x = y$.

Suppose $x \neq y$, their cubes will also not be equal.

$\Rightarrow x^3 \neq y^3$

However, this will be a contradiction to (1).

∴ $x = y$

Hence, *f* is injective.

Question 6:

Give examples of two functions f: **N** → **Z** and g: **Z** → **Z** such that $g \circ f$ is injective but g is not injective.

(Hint: Consider $f(x) = x$ and $g(x) = |x|$)

Answer

Define f: **N** → **Z** as $f(x) = x$ and g: **Z** → **Z** as $g(x) = |x|$.

We first show that *g* is not injective.

It can be observed that:

$g(-1) = |-1| = 1$

$g(1) = |1| = 1$

∴ $g(-1) = g(1)$, but $-1 \neq 1$.

∴ g is not injective.

Now, *gof*: **N** → **Z** is defined as $gof(x) = g(f(x)) = g(x) = |x|$.

Let $x, y \in$ **N** such that $gof(x) = gof(y)$.

$\Rightarrow |x| = |y|$

Since x and $y \in$ **N,** both are positive.

$\therefore |x| = |y| \Rightarrow x = y$

Hence, *gof* is injective

Question 7:

Given examples of two functions *f*: **N** → **N** and *g*: **N** → **N** such that *gof* is onto but *f* is not onto.

(Hint: Consider $f(x) = x + 1$ and $g(x) = \begin{cases} x-1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$

Answer

Define *f*: **N** → **N** by,

$f(x) = x + 1$

And, *g*: **N** → **N** by,

$$g(x) = \begin{cases} x-1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$$

We first show that *g* is not onto.

For this, consider element 1 in co-domain **N**. It is clear that this element is not an image of any of the elements in domain **N**.

∴ *f* is not onto.

Now, *gof*: **N** → **N** is defined by,

$$gof(x) = g(f(x)) = g(x+1) = (x+1) - 1 \quad [x \in \mathbf{N} \Rightarrow (x+1) > 1]$$
$$= x$$

Then, it is clear that for $y \in$ **N**, there exists $x = y \in$ **N** such that $gof(x) = y$.

Hence, *gof* is onto.

Question 8:

Given a non empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows:

For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X)? Justify you answer:

Answer

Since every set is a subset of itself, ARA for all $A \in$ P(X).

∴R is reflexive.

Let AR$B \Rightarrow A \subset B$.

This cannot be implied to $B \subset A$.

For instance, if A = {1, 2} and B = {1, 2, 3}, then it cannot be implied that B is related to A.

∴ R is not symmetric.

Further, if ARB and BRC, then $A \subset B$ and $B \subset C$.

$\Rightarrow A \subset C$

$\Rightarrow A$RC

∴ R is transitive.

Hence, R is not an equivalence relation since it is not symmetric.

Question 9:

Given a non-empty set X, consider the binary operation *: P(X) × P(X) → P(X) given by A * B = $A \cap B$ ☐ A, B in P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in P(X) with respect to the operation*.

Answer

It is given that $*: \mathrm{P}(X) \times \mathrm{P}(X) \to \mathrm{P}(X)$ is defined as $A * B = A \cap B \ \forall \ A, B \in \mathrm{P}(X)$.

We know that $A \cap X = A = X \cap A \ \forall \ A \in \mathrm{P}(X)$.

$\Rightarrow A * X = A = X * A \ \forall \ A \in \mathrm{P}(X)$

Thus, X is the identity element for the given binary operation *.

Now, an element $A \in \mathrm{P}(X)$ is invertible if there exists $B \in \mathrm{P}(X)$ such that

$A * B = X = B * A.$ (As X is the identity element)

i.e.,

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

Thus, X is the only invertible element in P(X) with respect to the given operation*.

Hence, the given result is proved.

Question 10:

Find the number of all onto functions from the set {1, 2, 3, … , n) to itself.

Answer

Onto functions from the set {1, 2, 3, … ,n} to itself is simply a permutation on n symbols 1, 2, …, n.

Thus, the total number of onto maps from {1, 2, … , n} to itself is the same as the total number of permutations on n symbols 1, 2, …, n, which is n.

Question 11:

Let S = {a, b, c} and T = {1, 2, 3}. Find F^{-1} of the following functions F from S to T, if it exists.

(i) F = {(a, 3), (b, 2), (c, 1)} (ii) F = {(a, 2), (b, 1), (c, 1)}

Answer

S = {a, b, c}, T = {1, 2, 3}

(i) F: $S \to T$ is defined as:

F = {(a, 3), (b, 2), (c, 1)}

⇒ F (a) = 3, F (b) = 2, F(c) = 1

Therefore, F^{-1}: $T \to S$ is given by

F^{-1} = {(3, a), (2, b), (1, c)}.

(ii) F: $S \to T$ is defined as:

F = {(a, 2), (b, 1), (c, 1)}

Since F (b) = F (c) = 1, F is not one-one.

Hence, F is not invertible i.e., F^{-1} does not exist.

Question 12:

Consider the binary operations*: **R** ×**R** → and o: **R** × **R** → **R** defined as $a * b = |a-b|$ and *a* o *b* = *a*, □*a*, *b* ∈ **R**. Show that * is commutative but not associative, o is associative but not commutative. Further, show that □*a*, *b*, *c* ∈ **R**, *a**(*b* o *c*) = (*a* * *b*) o (*a* * *c*). [If it is so, we say that the operation * distributes over the operation o]. Does o distribute over *? Justify your answer.

Answer

It is given that *: **R** ×**R** → and o: **R** × **R** → **R** isdefined as

$a * b = |a-b|$ and *a* o *b* = *a*, □*a*, *b* ∈ **R**.

For *a*, *b* ∈ **R**, we have:

$a * b = |a-b|$

$b * a = |b-a| = |-(a-b)| = |a-b|$

∴*a* * *b* = *b* * *a*

∴ The operation * is commutative.

It can be observed that,

$(1*2)*3 = (|1-2|)*3 = 1*3 = |1-3| = 2$

$1*(2*3) = 1*(|2-3|) = 1*1 = |1-1| = 0$

$\therefore (1*2)*3 \neq 1*(2*3)$ (where 1, 2, 3 ∈ **R**)

∴The operation * is not associative.

Now, consider the operation o:

It can be observed that 1 o 2 = 1 and 2 o 1 = 2.

∴1 o 2 ≠ 2 o 1 (where 1, 2 ∈ **R**)

∴The operation o is not commutative.

Let *a*, *b*, *c* ∈ **R**. Then, we have:

(*a* o *b*) o *c* = *a* o *c* = *a*

a o (*b* o *c*) = *a* o *b* = *a*

⇒ *a* o *b*) o c = *a* o (*b* o *c*)

∴ The operation o is associative.

Now, let *a*, *b*, *c* ∈ **R**, then we have:

a * (*b* o *c*) = a * *b* = $|a-b|$

(*a* * *b*) o (*a* * *c*) = $(|a-b|) \text{o} (|a-c|) = |a-b|$

Hence, *a* * (*b* o *c*) = (*a* * *b*) o (*a* * *c*).

Now,

1 o (2 * 3) = $1 \text{ o} (|2-3|) = 1 \text{ o } 1 = 1$

(1 o 2) * (1 o 3) = 1 * 1 = $|1-1| = 0$

∴1 o (2 * 3) ≠ (1 o 2) * (1 o 3) (where 1, 2, 3 ∈ **R**)

∴ The operation o does not distribute over *.

Question 13:

Given a non-empty set *X*, let *: P(*X*) × P(*X*) → P(*X*) be defined as *A* * *B* = (*A* − *B*) ∪ (*B* − *A*), □ *A*, *B* ∈ P(*X*). Show that the empty set *Φ* is the identity for the operation * and all the elements *A* of P(*X*) are invertible with $A^{-1} = A$. (Hint: (*A* − *Φ*) ∪ (*Φ* − *A*) = *A* and (*A* − *A*) ∪ (*A* − *A*) = *A* * *A* = *Φ*).

Answer

It is given that *: P(*X*) × P(*X*) → P(*X*) is defined as

A * *B* = (*A* − *B*) ∪ (*B* − *A*) □ *A*, *B* ∈ P(*X*).

Let *A* ∈ P(*X*). Then, we have:

A * *Φ* = (*A* − *Φ*) ∪ (*Φ* − *A*) = *A* ∪ *Φ* = *A*

Φ * *A* = (*Φ* − *A*) ∪ (*A* − *Φ*) = *Φ* ∪ *A* = *A*

∴*A* * *Φ* = *A* = *Φ* * *A*. □ *A* ∈ P(*X*)

Thus, *Φ* is the identity element for the given operation*.

Now, an element *A* ∈ P(*X*) will be invertible if there exists *B* ∈ P(*X*) such that

A * *B* = *Φ* = *B* * *A*. (As Φ is the identity element)

Now, we observed that $A * A = (A-A) \cup (A-A) = \phi \cup \phi = \phi \ \forall A \in \text{P}(X)$.

Hence, all the elements *A* of P(*X*) are invertible with $A^{-1} = A$.

Question 14:

Define a binary operation *on the set {0, 1, 2, 3, 4, 5} as

$$a*b=\begin{cases}a+b, & \text{if } a+b<6\\ a+b-6 & \text{if } a+b\geq 6\end{cases}$$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a.

Answer

Let $X = \{0, 1, 2, 3, 4, 5\}$.

The operation * on X is defined as:

$$a*b=\begin{cases}a+b & \text{if } a+b<6\\ a+b-6 & \text{if } a+b\geq 6\end{cases}$$

An element $e \in X$ is the identity element for the operation *, if $a*e=a=e*a\ \forall a\in X$.

For $a \in X$, we observed that:

$a*0=a+0=a \qquad [a\in X \Rightarrow a+0<6]$

$0*a=0+a=a \qquad [a\in X \Rightarrow 0+a<6]$

$\therefore a*0=a=0*a\ \forall a\in X$

Thus, 0 is the identity element for the given operation *.

An element $a \in X$ is invertible if there exists $b\in X$ such that $a * b = 0 = b * a$.

$$\text{i.e.,}\begin{cases}a+b=0=b+a, & \text{if } a+b<6\\ a+b-6=0=b+a-6, & \text{if } a+b\geq 6\end{cases}$$

i.e.,

$a = -b$ or $b = 6 - a$

But, $X = \{0, 1, 2, 3, 4, 5\}$ and $a, b \in X$. Then, $a \neq -b$.

$\therefore b = 6 - a$ is the inverse of a □ $a \in X$.

Hence, the inverse of an element $a \in X$, $a \neq 0$ is $6 - a$ i.e., $a^{-1} = 6 - a$.

Question 15:

Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g: A \to B$ be functions defined by $f(x) = x^2 - x, x \in A$ and $g(x)=2\left|x-\frac{1}{2}\right|-1,\ x\in A$. Are f and g equal?

Justify your answer. (Hint: One may note that two function $f: A \to B$ and $g: A \to B$ such that $f(a) = g(a)$ □$a \in A$, are called equal functions).

Answer

It is given that $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$.

Also, it is given that $f, g: A \to B$ are defined by $f(x) = x^2 - x$, $x \in A$ and

$$g(x) = 2\left|x - \frac{1}{2}\right| - 1,\ x \in A$$
.

It is observed that:

$$f(-1) = (-1)^2 - (-1) = 1 + 1 = 2$$

$$g(-1) = 2\left|(-1) - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(-1) = g(-1)$$

$$f(0) = (0)^2 - 0 = 0$$

$$g(0) = 2\left|0 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(0) = g(0)$$

$$f(1) = (1)^2 - 1 = 1 - 1 = 0$$

$$g(1) = 2\left|1 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0$$

$$\Rightarrow f(1) = g(1)$$

$$f(2) = (2)^2 - 2 = 4 - 2 = 2$$

$$g(2) = 2\left|2 - \frac{1}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2$$

$$\Rightarrow f(2) = g(2)$$

$$\therefore f(a) = g(a)\ \forall a \in A$$

Hence, the functions f and g are equal.

Question 16:

Let $A = \{1, 2, 3\}$. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4

Answer

The given set is A = {1, 2, 3}.

The smallest relation containing (1, 2) and (1, 3) which is reflexive and symmetric, but not transitive is given by:

R = {(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (3, 1)}

This is because relation R is reflexive as (1, 1), (2, 2), (3, 3) ∈ R.

Relation R is symmetric since (1, 2), (2, 1) ∈R and (1, 3), (3, 1) ∈R.

But relation R is not transitive as (3, 1), (1, 2) ∈ R, but (3, 2) ∉ R.

Now, if we add any two pairs (3, 2) and (2, 3) (or both) to relation R, then relation R will become transitive.

Hence, the total number of desired relations is one.

The correct answer is A.

Question 17:

Let *A* = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4

Answer

It is given that *A* = {1, 2, 3}.

The smallest equivalence relation containing (1, 2) is given by,

R_1 = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)}

Now, we are left with only four pairs i.e., (2, 3), (3, 2), (1, 3), and (3, 1).

If we odd any one pair [say (2, 3)] to R_1, then for symmetry we must add (3, 2). Also, for transitivity we are required to add (1, 3) and (3, 1).

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing (1, 2) is two.

The correct answer is B.

Question 18:

Let *f*: **R** → **R** be the Signum Function defined as

$$f(x)=\begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$$

and *g*: **R** → **R** be the Greatest Integer Function given by *g*(*x*) = [*x*], where [*x*] is greatest integer less than or equal to *x*. Then does *fog* and *gof* coincide in (0, 1]?

Answer

It is given that,

f: **R** → **R** is defined as $f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$

Also, g: **R** → **R** is defined as $g(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x.

Now, let $x \in (0, 1]$.

Then, we have:

$[x] = 1$ if $x = 1$ and $[x] = 0$ if $0 < x < 1$.

$$\therefore fog(x)=f(g(x))=f([x])=\begin{cases}f(1), & \text{if } x=1\\ f(0), & \text{if } x\in(0,1)\end{cases}=\begin{cases}1, & \text{if } x=1\\ 0, & \text{if } x\in(0,1)\end{cases}$$

$$\begin{aligned}gof(x)&=g(f(x))\\&=g(1) \qquad [x>0]\\&=[1]=1\end{aligned}$$

Thus, when $x \in (0, 1)$, we have $fog(x) = 0$and $gof(x) = 1$.

Hence, *fog* and *gof* do not coincide in (0, 1].

Question 19:

Number of binary operations on the set {*a*, *b*} are

(A) 10 (B) 16 (C) 20 (D) 8

Answer

A binary operation * on {*a*, *b*} is a function from {*a*, *b*} × {*a*, *b*} → {*a*, *b*}

i.e., * is a function from {(*a*, *a*), (*a*, *b*), (*b*, *a*), (*b*, *b*)} → {*a*, *b*}.

Hence, the total number of binary operations on the set {*a*, *b*} is 2^4 i.e., 16.

The correct answer is B.