Exercise 1.1

Question 1:

Determine whether each of the following relations are reflexive, symmetric and transitive:

(i)Relation R in the set $A = \{1, 2, 3 \ldots 13, 14\}$ defined as

$R = \{(x, y): 3x - y = 0\}$

(ii) Relation R in the set **N** of natural numbers defined as

$R = \{(x, y): y = x + 5 \text{ and } x < 4\}$

(iii) Relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ as

$R = \{(x, y): y$ is divisible by $x\}$

(iv) Relation R in the set **Z** of all integers defined as

$R = \{(x, y): x - y$ is as integer$\}$

(v) Relation R in the set A of human beings in a town at a particular time given by

(a) $R = \{(x, y): x$ and y work at the same place$\}$

(b) $R = \{(x, y): x$ and y live in the same locality$\}$

(c) $R = \{(x, y): x$ is exactly 7 cm taller than $y\}$

(d) $R = \{(x, y): x$ is wife of $y\}$

(e) $R = \{(x, y): x$ is father of $y\}$

Answer

(i) $A = \{1, 2, 3 \ldots 13, 14\}$

$R = \{(x, y): 3x - y = 0\}$

$\therefore R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$

R is not reflexive since $(1, 1), (2, 2) \ldots (14, 14) \notin R$.

Also, R is not symmetric as $(1, 3) \in R$, but $(3, 1) \notin R$. $[3(3) - 1 \neq 0]$

Also, R is not transitive as $(1, 3), (3, 9) \in R$, but $(1, 9) \notin R$.

$[3(1) - 9 \neq 0]$

Hence, R is neither reflexive, nor symmetric, nor transitive.

(ii) $R = \{(x, y): y = x + 5 \text{ and } x < 4\} = \{(1, 6), (2, 7), (3, 8)\}$

It is seen that $(1, 1) \notin R$.

$\therefore R$ is not reflexive.

$(1, 6) \in R$

But,

(1, 6) ∉ R.

∴R is not symmetric.

Now, since there is no pair in R such that (x, y) and (y, z) ∈R, then (x, z) cannot belong to R.

∴ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(iii) A = {1, 2, 3, 4, 5, 6}

R = {(x, y): y is divisible by x}

We know that any number (x) is divisible by itself.

⇒(x, x) ∈R

∴R is reflexive.

Now,

(2, 4) ∈R [as 4 is divisible by 2]

But,

(4, 2) ∉ R. [as 2 is not divisible by 4]

∴R is not symmetric.

Let (x, y), (y, z) ∈ R. Then, y is divisible by x and z is divisible by y.

∴z is divisible by x.

⇒ (x, z) ∈R

∴R is transitive.

Hence, R is reflexive and transitive but not symmetric.

(iv) R = {(x, y): $x - y$ is an integer}

Now, for every x ∈ **Z**, (x, x) ∈R as $x - x = 0$ is an integer.

∴R is reflexive.

Now, for every x, y ∈ **Z** if (x, y) ∈ R, then $x - y$ is an integer.

⇒ $-(x - y)$ is also an integer.

⇒ $(y - x)$ is an integer.

∴ (y, x) ∈ R

∴R is symmetric.

Now,

Let (x, y) and (y, z) ∈R, where x, y, z ∈ **Z**.

⇒ $(x - y)$ and $(y - z)$ are integers.

⇒ $x - z = (x - y) + (y - z)$ is an integer.

∴ $(x, z) \in R$

∴R is transitive.

Hence, R is reflexive, symmetric, and transitive.

(v) (a) R = {(x, y): x and y work at the same place}

⇒ $(x, x) \in R$

∴ R is reflexive.

If $(x, y) \in R$, then x and y work at the same place.

⇒ y and x work at the same place.

⇒ $(y, x) \in R$.

∴R is symmetric.

Now, let (x, y), $(y, z) \in R$

⇒ x and y work at the same place and y and z work at the same place.

⇒ x and z work at the same place.

⇒ $(x, z) \in R$

∴ R is transitive.

Hence, R is reflexive, symmetric, and transitive.

(b) R = {(x, y): x and y live in the same locality}

Clearly $(x, x) \in R$ as x and x is the same human being.

∴ R is reflexive.

If $(x, y) \in R$, then x and y live in the same locality.

⇒ y and x live in the same locality.

⇒ $(y, x) \in R$

∴R is symmetric.

Now, let $(x, y) \in R$ and $(y, z) \in R$.

⇒ x and y live in the same locality and y and z live in the same locality.

⇒ x and z live in the same locality.

⇒ $(x, z) \in R$

∴ R is transitive.

Hence, R is reflexive, symmetric, and transitive.

(c) R = {(x, y): x is exactly 7 cm taller than y}

Now,

$(x, x) \notin R$

Since human being x cannot be taller than himself.

∴R is not reflexive.

Now, let $(x, y) \in$R.

⇒ x is exactly 7 cm taller than y.

Then, y is not taller than x.

∴ $(y, x) \notin$R

Indeed if x is exactly 7 cm taller than y, then y is exactly 7 cm shorter than x.

∴R is not symmetric.

Now,

Let (x, y), $(y, z) \in$ R.

⇒ x is exactly 7 cm taller than y and y is exactly 7 cm taller than z.

⇒ x is exactly 14 cm taller than z .

∴$(x, z) \notin$R

∴ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(d) R = $\{(x, y)$: x is the wife of $y\}$

Now,

$(x, x) \notin$ R

Since x cannot be the wife of herself.

∴R is not reflexive.

Now, let $(x, y) \in$ R

⇒ x is the wife of y.

Clearly y is not the wife of x.

∴$(y, x) \notin$ R

Indeed if x is the wife of y, then y is the husband of x.

∴ R is not transitive.

Let (x, y), $(y, z) \in$ R

⇒ x is the wife of y and y is the wife of z.

This case is not possible. Also, this does not imply that x is the wife of z.

∴$(x, z) \notin$ R

∴R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

(e) R = $\{(x, y)$: x is the father of $y\}$

$(x, x) \notin$ R

As x cannot be the father of himself.

∴R is not reflexive.

Now, let $(x, y) \in R$.

⇒ x is the father of y.

⇒ y cannot be the father of y.

Indeed, y is the son or the daughter of y.

∴$(y, x) \notin R$

∴ R is not symmetric.

Now, let $(x, y) \in R$ and $(y, z) \in R$.

⇒ x is the father of y and y is the father of z.

⇒ x is not the father of z.

Indeed x is the grandfather of z.

∴ $(x, z) \notin R$

∴R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 2:

Show that the relation R in the set **R** of real numbers, defined as

$R = \{(a, b): a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.

Answer

$R = \{(a, b): a \leq b^2\}$

It can be observed that $\left(\frac{1}{2}, \frac{1}{2}\right) \notin \mathbf{R}$, since $\frac{1}{2} > \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

∴R is not reflexive.

Now, $(1, 4) \in R$ as $1 < 4^2$

But, 4 is not less than 1^2.

∴$(4, 1) \notin R$

∴R is not symmetric.

Now,

$(3, 2), (2, 1.5) \in R$

(as $3 < 2^2 = 4$ and $2 < (1.5)^2 = 2.25$)

But, $3 > (1.5)^2 = 2.25$

∴(3, 1.5) ∉ R

∴ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 3:

Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as

R = {(*a*, *b*): *b* = *a* + 1} is reflexive, symmetric or transitive.

Answer

Let *A* = {1, 2, 3, 4, 5, 6}.

A relation R is defined on set *A* as:

R = {(*a*, *b*): *b* = *a* + 1}

∴R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}

We can find (*a*, *a*) ∉ R, where *a* ∈ A.

For instance,

(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6) ∉ R

∴R is not reflexive.

It can be observed that (1, 2) ∈ R, but (2, 1) ∉ R.

∴R is not symmetric.

Now, (1, 2), (2, 3) ∈ **R**

But,

(1, 3) ∉ R

∴R is not transitive

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 4:

Show that the relation R in **R** defined as R = {(*a*, *b*): *a* ≤ *b*}, is reflexive and transitive but not symmetric.

Answer

R = {(*a*, *b*); *a* ≤ *b*}

Clearly (*a*, *a*) ∈ R as *a* = *a*.

∴R is reflexive.

Now,

(2, 4) ∈ R (as 2 < 4)

But, (4, 2) ∉ R as 4 is greater than 2.

∴ R is not symmetric.

Now, let $(a, b), (b, c) \in R$.

Then,

$a \le b$ and $b \le c$

$\Rightarrow a \le c$

$\Rightarrow (a, c) \in R$

∴R is transitive.

Hence,R is reflexive and transitive but not symmetric.

Question 5:

Check whether the relation R in **R** defined as $R = \{(a, b): a \le b^3\}$ is reflexive, symmetric or transitive.

Answer

$R = \{(a, b): a \le b^3\}$

It is observed that $\left(\frac{1}{2}, \frac{1}{2}\right) \notin R$ as $\frac{1}{2} > \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

∴ R is not reflexive.

Now,

$(1, 2) \in R$ (as $1 < 2^3 = 8$)

But,

$(2, 1) \notin R$ (as $2^3 > 1$)

∴ R is not symmetric.

We have $\left(3, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{6}{5}\right) \in R$ as $3 < \left(\frac{3}{2}\right)^3$ and $\frac{3}{2} < \left(\frac{6}{5}\right)^3$.

But $\left(3, \frac{6}{5}\right) \notin R$ as $3 > \left(\frac{6}{5}\right)^3$.

∴ R is not transitive.

Hence, R is neither reflexive, nor symmetric, nor transitive.

Question 6:

Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

Answer

Let A = {1, 2, 3}.

A relation R on A is defined as R = {(1, 2), (2, 1)}.

It is seen that (1, 1), (2, 2), (3, 3) ∉R.

∴ R is not reflexive.

Now, as (1, 2) ∈ R and (2, 1) ∈ R, then R is symmetric.

Now, (1, 2) and (2, 1) ∈ R

However,

(1, 1) ∉ R

∴ R is not transitive.

Hence, R is symmetric but neither reflexive nor transitive.

Question 7:

Show that the relation R in the set A of all the books in a library of a college, given by R = {(x, y): x and y have same number of pages} is an equivalence relation.

Answer

Set A is the set of all books in the library of a college.

R = {x, y): x and y have the same number of pages}

Now, R is reflexive since (x, x) ∈ R as x and x has the same number of pages.

Let (x, y) ∈ R ⇒ x and y have the same number of pages.

⇒ y and x have the same number of pages.

⇒ (y, x) ∈ R

∴R is symmetric.

Now, let (x, y) ∈R and (y, z) ∈ R.

⇒ x and y and have the same number of pages and y and z have the same number of pages.

⇒ x and z have the same number of pages.

⇒ (x, z) ∈ R

∴R is transitive.

Hence, R is an equivalence relation.

Question 8:

Show that the relation R in the set A = {1, 2, 3, 4, 5} given by $R = \{(a, b) : |a-b| \text{ is even}\}$, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of 2, 4}.

Answer

A = {1, 2, 3, 4, 5}

$R = \{(a, b) : |a-b| \text{ is even}\}$

It is clear that for any element $a \in A$, we have $|a-a| = 0$ (which is even).

∴R is reflexive.

Let $(a, b) \in R$.

$\Rightarrow |a-b| \text{ is even.}$

$\Rightarrow |-(a-b)| = |b-a| \text{ is also even.}$

$\Rightarrow (b, a) \in R$

∴R is symmetric.

Now, let $(a, b) \in R$ and $(b, c) \in R$.

$\Rightarrow |a-b| \text{ is even and } |b-c| \text{ is even.}$

$\Rightarrow (a-b) \text{ is even and } (b-c) \text{ is even.}$

$\Rightarrow (a-c) = (a-b)+(b-c) \text{ is even.}$ [Sum of two even integers is even]

$\Rightarrow |a-c| \text{ is even.}$

$\Rightarrow (a, c) \in R$

∴R is transitive.

Hence, R is an equivalence relation.

Now, all elements of the set {1, 2, 3} are related to each other as all the elements of this subset are odd. Thus, the modulus of the difference between any two elements will be even.

Similarly, all elements of the set {2, 4} are related to each other as all the elements of this subset are even.

Also, no element of the subset {1, 3, 5} can be related to any element of {2, 4} as all elements of {1, 3, 5} are odd and all elements of {2, 4} are even. Thus, the modulus of the difference between the two elements (from each of these two subsets) will not be even.

Question 9:

Show that each of the relation R in the set $A = \{x \in \mathbf{Z} : 0 \le x \le 12\}$, given by

(i) $R = \{(a, b) : |a-b| \text{ is a multiple of } 4\}$

(ii) $R = \{(a, b) : a = b\}$

is an equivalence relation. Find the set of all elements related to 1 in each case.

Answer

$A = \{x \in \mathbf{Z} : 0 \le x \le 12\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

(i) $R = \{(a, b) : |a-b| \text{ is a multiple of } 4\}$

For any element *a* ∈A, we have (*a*, *a*) ∈ R as $|a-a| = 0$ is a multiple of 4.

∴R is reflexive.

Now, let (*a*, *b*) ∈ R ⇒ $|a-b|$ is a multiple of 4.

$\Rightarrow |-(a-b)| = |b-a|$ is a multiple of 4.

⇒ (*b*, *a*) ∈ R

∴R is symmetric.

Now, let (*a*, *b*), (*b*, *c*) ∈ R.

$\Rightarrow |a-b|$ is a multiple of 4 and $|b-c|$ is a multiple of 4.

$\Rightarrow (a-b)$ is a multiple of 4 and $(b-c)$ is a multiple of 4.

$\Rightarrow (a-c) = (a-b)+(b-c)$ is a multiple of 4.

$\Rightarrow |a-c|$ is a multiple of 4.

⇒ (*a*, *c*) ∈R

∴ R is transitive.

Hence, R is an equivalence relation.

The set of elements related to 1 is {1, 5, 9} since

$|1-1| = 0$ is a multiple of 4,

$|5-1| = 4$ is a multiple of 4, and

$|9-1| = 8$ is a multiple of 4.

(ii) R = {(a, b): $a = b$}

For any element $a \in$A, we have (a, a) ∈ R, since $a = a$.

∴R is reflexive.

Now, let (a, b) ∈ R.

⇒ $a = b$

⇒ $b = a$

⇒ (b, a) ∈ R

∴R is symmetric.

Now, let (a, b) ∈ R and (b, c) ∈ R.

⇒ $a = b$ and $b = c$

⇒ $a = c$

⇒ (a, c) ∈ R

∴ R is transitive.

Hence, R is an equivalence relation.

The elements in R that are related to 1 will be those elements from set A which are equal to 1.

Hence, the set of elements related to 1 is {1}.

Question 10:

Given an example of a relation. Which is

(i) Symmetric but neither reflexive nor transitive.

(ii) Transitive but neither reflexive nor symmetric.

(iii) Reflexive and symmetric but not transitive.

(iv) Reflexive and transitive but not symmetric.

(v) Symmetric and transitive but not reflexive.

Answer

(i) Let A = {5, 6, 7}.

Define a relation R on A as R = {(5, 6), (6, 5)}.

Relation R is not reflexive as (5, 5), (6, 6), (7, 7) ∉ R.

Now, as (5, 6) ∈ R and also (6, 5) ∈ R, R is symmetric.

⇒ (5, 6), (6, 5) ∈ R, but (5, 5) ∉ R

∴R is not transitive.

Hence, relation R is symmetric but not reflexive or transitive.

(ii)Consider a relation R in **R** defined as:

R = {(a, b): $a < b$}

For any a ∈ R, we have (a, a) ∉ R since a cannot be strictly less than a itself. In fact, a = a.

∴ R is not reflexive.

Now,

(1, 2) ∈ R (as 1 < 2)

But, 2 is not less than 1.

∴ (2, 1) ∉ R

∴ R is not symmetric.

Now, let (a, b), (b, c) ∈ R.

⇒ $a < b$ and $b < c$

⇒ $a < c$

⇒ (a, c) ∈ R

∴R is transitive.

Hence, relation R is transitive but not reflexive and symmetric.

(iii)Let A = {4, 6, 8}.

Define a relation R on A as:

A = {(4, 4), (6, 6), (8, 8), (4, 6), (6, 4), (6, 8), (8, 6)}

Relation R is reflexive since for every $a \in A$, (a, a) ∈R i.e., (4, 4), (6, 6), (8, 8)} ∈ R.

Relation R is symmetric since (a, b) ∈ R ⇒ (b, a) ∈ R for all a, b ∈ R.

Relation R is not transitive since (4, 6), (6, 8) ∈ R, but (4, 8) ∉ R.

Hence, relation R is reflexive and symmetric but not transitive.

(iv) Define a relation R in **R** as:

R = {a, b): $a^3 \geq b^3$}

Clearly (a, a) ∈ R as $a^3 = a^3$.

∴R is reflexive.

Now,

(2, 1) ∈ R (as $2^3 \geq 1^3$)

But,

(1, 2) ∉ R (as $1^3 < 2^3$)

∴ R is not symmetric.

Now,

Let (a, b), (b, c) ∈ R.

⇒ $a^3 \geq b^3$ and $b^3 \geq c^3$

⇒ $a^3 \geq c^3$

⇒ (a, c) ∈ R

∴R is transitive.

Hence, relation R is reflexive and transitive but not symmetric.

(v) Let A = {−5, −6}.

Define a relation R on A as:

R = {(−5, −6), (−6, −5), (−5, −5)}

Relation R is not reflexive as (−6, −6) ∉ R.

Relation R is symmetric as (−5, −6) ∈ R and (−6, −5}∈R.

It is seen that (−5, −6), (−6, −5) ∈ R. Also, (−5, −5) ∈ R.

∴The relation R is transitive.

Hence, relation R is symmetric and transitive but not reflexive.

Question 11:

Show that the relation R in the set A of points in a plane given by R = {(P, Q): distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all point related to a point P ≠ (0, 0) is the circle passing through P with origin as centre.

Answer

R = {(P, Q): distance of point P from the origin is the same as the distance of point Q from the origin}

Clearly, (P, P) ∈ R since the distance of point P from the origin is always the same as the distance of the same point P from the origin.

∴R is reflexive.

Now,

Let (P, Q) ∈ R.

⇒ The distance of point P from the origin is the same as the distance of point Q from the origin.

⇒ The distance of point Q from the origin is the same as the distance of point P from the origin.

⇒ (Q, P) ∈ R

∴R is symmetric.

Now,

Let (P, Q), (Q, S) ∈ R.

⇒ The distance of points P and Q from the origin is the same and also, the distance of points Q and S from the origin is the same.

⇒ The distance of points P and S from the origin is the same.

⇒ (P, S) ∈ R

∴R is transitive.

Therefore, R is an equivalence relation.

The set of all points related to P ≠ (0, 0) will be those points whose distance from the origin is the same as the distance of point P from the origin.

In other words, if O (0, 0) is the origin and OP = k, then the set of all points related to P is at a distance of k from the origin.

Hence, this set of points forms a circle with the centre as the origin and this circle passes through point P.

Question 12:

Show that the relation R defined in the set A of all triangles as R = {(T_1, T_2): T_1 is similar to T_2}, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related?

Answer

R = {(T_1, T_2): T_1 is similar to T_2}

R is reflexive since every triangle is similar to itself.

Further, if (T_1, T_2) ∈ R, then T_1 is similar to $T_{2.}$

⇒ T_2 is similar to $T_{1.}$

⇒ (T_2, T_1) ∈R

∴R is symmetric.

Now,

Let (T_1, T_2), $(T_2, T_3) \in$ R.

⇒ T_1 is similar to T_2 and T_2 is similar to T_3.

⇒ T_1 is similar to T_3.

⇒ $(T_1, T_3) \in$ R

∴ R is transitive.

Thus, R is an equivalence relation.

Now, we can observe that:

$$\frac{3}{6}=\frac{4}{8}=\frac{5}{10}\left(=\frac{1}{2}\right)$$

∴The corresponding sides of triangles T_1 and T_3 are in the same ratio.

Then, triangle T_1 is similar to triangle T_3.

Hence, T_1 is related to T_3.

Question 13:

Show that the relation R defined in the set *A* of all polygons as R = $\{(P_1, P_2)$: P_1 and P_2 have same number of sides$\}$, is an equivalence relation. What is the set of all elements in *A* related to the right angle triangle *T* with sides 3, 4 and 5?

Answer

R = $\{(P_1, P_2)$: P_1 and P_2 have same the number of sides$\}$

R is reflexive since $(P_1, P_1) \in$ R as the same polygon has the same number of sides with itself.

Let $(P_1, P_2) \in$ R.

⇒ P_1 and P_2 have the same number of sides.

⇒ P_2 and P_1 have the same number of sides.

⇒ $(P_2, P_1) \in$ R

∴R is symmetric.

Now,

Let (P_1, P_2), $(P_2, P_3) \in$ R.

⇒ P_1 and P_2 have the same number of sides. Also, P_2 and P_3 have the same number of sides.

⇒ P_1 and P_3 have the same number of sides.

⇒ $(P_1, P_3) \in$ R

∴R is transitive.

Hence, R is an equivalence relation.

The elements in A related to the right-angled triangle (T) with sides 3, 4, and 5 are those polygons which have 3 sides (since T is a polygon with 3 sides).

Hence, the set of all elements in A related to triangle T is the set of all triangles.

Question 14:

Let L be the set of all lines in XY plane and R be the relation in L defined as R = $\{(L_1, L_2): L_1$ is parallel to $L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

Answer

R = $\{(L_1, L_2)$: L_1 is parallel to $L_2\}$

R is reflexive as any line L_1 is parallel to itself i.e., $(L_1, L_1) \in$ R.

Now,

Let $(L_1, L_2) \in$ R.

$\Rightarrow L_1$ is parallel to L_2.

$\Rightarrow L_2$ is parallel to L_1.

$\Rightarrow (L_2, L_1) \in$ R

∴ R is symmetric.

Now,

Let $(L_1, L_2), (L_2, L_3) \in$R.

$\Rightarrow L_1$ is parallel to L_2. Also, L_2 is parallel to L_3.

$\Rightarrow L_1$ is parallel to L_3.

∴R is transitive.

Hence, R is an equivalence relation.

The set of all lines related to the line $y = 2x + 4$ is the set of all lines that are parallel to the line $y = 2x + 4$.

Slope of line $y = 2x + 4$ is $m = 2$

It is known that parallel lines have the same slopes.

The line parallel to the given line is of the form $y = 2x + c$, where $c \in \mathbf{R}$.

Hence, the set of all lines related to the given line is given by $y = 2x + c$, where $c \in \mathbf{R}$.

Question 15:

Let R be the relation in the set {1, 2, 3, 4} given by R = {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}. Choose the correct answer.

(A) R is reflexive and symmetric but not transitive.

(B) R is reflexive and transitive but not symmetric.

(C) R is symmetric and transitive but not reflexive.

(D) R is an equivalence relation.

Answer

R = {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}

It is seen that $(a, a) \in$ **R,** for every $a \in\{1, 2, 3, 4\}$.

∴ R is reflexive.

It is seen that (1, 2) ∈ R, but (2, 1) ∉ R.

∴R is not symmetric.

Also, it is observed that $(a, b), (b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in \{1, 2, 3, 4\}$.

∴ R is transitive.

Hence, R is reflexive and transitive but not symmetric.

The correct answer is B.

Question 16:

Let R be the relation in the set **N** given by $R = \{(a, b): a = b - 2, b > 6\}$. Choose the correct answer.

(A) (2, 4) ∈ R (B) (3, 8) ∈R (C) (6, 8) ∈R (D) (8, 7) ∈ R

Answer

$R = \{(a, b): a = b - 2, b > 6\}$

Now, since $b > 6$, (2, 4) ∉ R

Also, as 3 ≠ 8 − 2, (3, 8) ∉ R

And, as 8 ≠ 7 − 2

∴ (8, 7) ∉ R

Now, consider (6, 8).

We have 8 > 6 and also, 6 = 8 − 2.

∴(6, 8) ∈ R

The correct answer is C.

Exercise 1.2

Question 1:

Show that the function *f*: **R**$_*$ → **R**$_*$ defined by $f(x)=\frac{1}{x}$ is one-one and onto, where **R**$_*$ is the set of all non-zero real numbers. Is the result true, if the domain **R**$_*$ is replaced by **N** with co-domain being same as **R**$_*$?

Answer

It is given that *f*: **R**$_*$ → **R**$_*$ is defined by $f(x)=\frac{1}{x}.$

One-one:

$$f(x)=f(y)$$
$$\Rightarrow \frac{1}{x}=\frac{1}{y}$$
$$\Rightarrow x=y$$

∴*f* is one-one.

Onto:

It is clear that for *y*∈ **R**$_*$, there exists $x=\frac{1}{y}\in R_*$ $(\text{Exists as } y\neq 0)$ such that

$$f(x)=\frac{1}{\left(\frac{1}{y}\right)}=y.$$

∴*f* is onto.

Thus, the given function *(f)* is one-one and onto.

Now, consider function *g*: **N** → **R**$_*$defined by

$$g(x)=\frac{1}{x}.$$

We have,

$$g(x_1)=g(x_2)\Rightarrow \frac{1}{x_1}=\frac{1}{x_2}\Rightarrow x_1=x_2$$

∴*g* is one-one.

Further, it is clear that g is not onto as for $1.2 \in \mathbf{R}_*$ there does not exit any x in **N** such that $g(x) = \frac{1}{1.2}$.

Hence, function g is one-one but not onto.

Question 2:

Check the injectivity and surjectivity of the following functions:

(i) f: $\mathbf{N} \rightarrow \mathbf{N}$ given by $f(x) = x^2$

(ii) f: $\mathbf{Z} \rightarrow \mathbf{Z}$ given by $f(x) = x^2$

(iii) f: $\mathbf{R} \rightarrow \mathbf{R}$ given by $f(x) = x^2$

(iv) f: $\mathbf{N} \rightarrow \mathbf{N}$ given by $f(x) = x^3$

(v) f: $\mathbf{Z} \rightarrow \mathbf{Z}$ given by $f(x) = x^3$

Answer

(i) f: $\mathbf{N} \rightarrow \mathbf{N}$ is given by,

$f(x) = x^2$

It is seen that for $x, y \in \mathbf{N}$, $f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in \mathbf{N}$. But, there does not exist any x in **N** such that $f(x) = x^2 = 2$.

$\therefore$ f is not surjective.

Hence, function f is injective but not surjective.

(ii) f: $\mathbf{Z} \rightarrow \mathbf{Z}$ is given by,

$f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore$ f is not injective.

Now, $-2 \in \mathbf{Z}$. But, there does not exist any element $x \in \mathbf{Z}$ such that $f(x) = x^2 = -2$.

$\therefore$ f is not surjective.

Hence, function f is neither injective nor surjective.

(iii) f: $\mathbf{R} \rightarrow \mathbf{R}$ is given by,

$f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore$ f is not injective.

Now, $-2 \in \mathbf{R}$. But, there does not exist any element $x \in \mathbf{R}$ such that $f(x) = x^2 = -2$.

∴ f is not surjective.

Hence, function f is neither injective nor surjective.

(iv) f: **N** → **N** given by,

$f(x) = x^3$

It is seen that for x, y ∈**N**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

∴f is injective.

Now, 2 ∈ **N**. But, there does not exist any element x in domain **N** such that $f(x) = x^3 = 2$.

∴ f is not surjective

Hence, function f is injective but not surjective.

(v) f: **Z** → **Z** is given by,

$f(x) = x^3$

It is seen that for x, y ∈ **Z**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

∴ f is injective.

Now, 2 ∈ **Z**. But, there does not exist any element x in domain **Z** such that $f(x) = x^3 = 2$.

∴ f is not surjective.

Hence, function f is injective but not surjective.

Question 3:

Prove that the Greatest Integer Function f: **R** → **R** given by $f(x) = [x]$, is neither one-once nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

Answer

f: **R** → **R** is given by,

$f(x) = [x]$

It is seen that $f(1.2) = [1.2] = 1$, $f(1.9) = [1.9] = 1$.

∴ $f(1.2) = f(1.9)$, but $1.2 \neq 1.9$.

∴ f is not one-one.

Now, consider 0.7 ∈ **R**.

It is known that $f(x) = [x]$ is always an integer. Thus, there does not exist any element x ∈ **R** such that $f(x) = 0.7$.

∴ f is not onto.

Hence, the greatest integer function is neither one-one nor onto.

Question 4:

Show that the Modulus Function *f*: **R** → **R** given by $f(x)=|x|$, is neither one-one nor onto, where $|x|$ is *x*, if *x* is positive or 0 and $|x|$ is − *x*, if *x* is negative.

Answer

f: **R** → **R** is given by,

$$f(x)=|x|=\begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

It is seen that $f(-1)=|-1|=1, f(1)=|1|=1$.

∴*f*(−1) = *f*(1), but −1 ≠ 1.

∴ *f* is not one-one.

Now, consider −1 ∈ **R**.

It is known that *f*(*x*) = $|x|$ is always non-negative. Thus, there does not exist any element *x* in domain **R** such that *f*(*x*) = $|x|$ = −1.

∴ *f* is not onto.

Hence, the modulus function is neither one-one nor onto.

Question 5:

Show that the Signum Function *f*: **R** → **R**, given by

$$f(x)=\begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is neither one-one nor onto.

Answer

f: **R** → **R** is given by,

$$f(x)=\begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

It is seen that *f*(1) = *f*(2) = 1, but 1 ≠ 2.

∴*f* is not one-one.

Now, as $f(x)$ takes only 3 values (1, 0, or −1) for the element −2 in co-domain **R**, there does not exist any x in domain **R** such that $f(x) = -2$.

∴ f is not onto.

Hence, the signum function is neither one-one nor onto.

Question 6:

Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that f is one-one.

Answer

It is given that $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$.

$f: A \to B$ is defined as $f = \{(1, 4), (2, 5), (3, 6)\}$.

∴ $f(1) = 4$, $f(2) = 5$, $f(3) = 6$

It is seen that the images of distinct elements of A under f are distinct.

Hence, function f is one-one.

Question 7:

In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) f: **R** → **R** defined by $f(x) = 3 - 4x$

(ii) f: **R** → **R** defined by $f(x) = 1 + x^2$

Answer

(i) f: **R** → **R** is defined as $f(x) = 3 - 4x$.

Let $x_1,\ x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2)$.

$$\Rightarrow 3-4x_1 = 3-4x_2$$
$$\Rightarrow -4x_1 = -4x_2$$
$$\Rightarrow x_1 = x_2$$

∴ f is one-one.

For any real number (y) in **R**, there exists $\frac{3-y}{4}$ in **R** such that

$$f\left(\frac{3-y}{4}\right) = 3-4\left(\frac{3-y}{4}\right) = y.$$

∴ f is onto.

Hence, *f* is bijective.

(ii) *f*: **R** → **R** is defined as

$f(x)=1+x^2$.

Let $x_1,\ x_2 \in \mathbf{R}$ such that $f(x_1)=f(x_2)$.

$\Rightarrow 1+x_1^2=1+x_2^2$
$\Rightarrow x_1^2=x_2^2$
$\Rightarrow x_1=\pm x_2$

∴ $f(x_1)=f(x_2)$ does not imply that $x_1=x_2$.

For instance,

$f(1)=f(-1)=2$

∴ *f* is not one-one.

Consider an element −2 in co-domain **R**.

It is seen that $f(x)=1+x^2$ is positive for all $x \in \mathbf{R}$.

Thus, there does not exist any *x* in domain **R** such that *f*(*x*) = −2.

∴ *f* is not onto.

Hence, *f* is neither one-one nor onto.

Question 8:

Let *A* and *B* be sets. Show that *f*: *A* × *B* → *B* × *A* such that (*a*, *b*) = (*b*, *a*) is bijective function.

Answer

f: *A* × *B* → *B* × *A* is defined as *f*(*a*, *b*) = (*b*, *a*).

Let $(a_1,\ b_1),\ (a_2,\ b_2) \in \mathrm{A}\times\mathrm{B}$ such that $f(a_1,\ b_1)=f(a_2,\ b_2)$.

$\Rightarrow (b_1,\ a_1)=(b_2,\ a_2)$
$\Rightarrow b_1=b_2$ and $a_1=a_2$
$\Rightarrow (a_1,\ b_1)=(a_2,\ b_2)$

∴ *f* is one-one.

Now, let (*b*, *a*) ∈ *B* × *A* be any element.

Then, there exists (*a*, *b*) ∈*A* × *B* such that *f*(*a*, *b*) = (*b*, *a*). [By definition of *f*]

∴ f is onto.

Hence, f is bijective.

Question 9:

Let f: **N** → **N** be defined by $f(n)=\begin{cases}\frac{n+1}{2}, & \text{if } n \text{ is odd}\\ \frac{n}{2}, & \text{if } n \text{ is even}\end{cases}$ for all $n \in \mathbf{N}$.

State whether the function f is bijective. Justify your answer.

Answer

f: **N** → **N** is defined as $f(n)=\begin{cases}\frac{n+1}{2}, & \text{if } n \text{ is odd}\\ \frac{n}{2}, & \text{if } n \text{ is even}\end{cases}$ for all $n \in \mathbf{N}$.

It can be observed that:

$$f(1)=\frac{1+1}{2}=1 \text{ and } f(2)=\frac{2}{2}=1 \qquad [\text{By definition of } f]$$

$$\therefore f(1)=f(2), \text{ where } 1 \neq 2.$$

∴ f is not one-one.

Consider a natural number *(n)* in co-domain **N.**

Case **I:** n is odd

∴$n = 2r + 1$ for some $r \in$ **N.** Then, there exists $4r + 1 \in$**N** such that

$$f(4r+1)=\frac{4r+1+1}{2}=2r+1.$$

Case **II:** n is even

∴$n = 2r$ for some $r \in$ **N.** Then,there exists $4r \in$**N** such that $f(4r)=\frac{4r}{2}=2r$.

∴ f is onto.

Hence, f is not a bijective function.

Question 10:

Let A = **R** − {3} and B = **R** − {1}. Consider the function f: A → B defined by

$f(x)=\left(\frac{x-2}{x-3}\right)$. Is *f* one-one and onto? Justify your answer.

Answer

A = **R** − {3}, B = **R** − {1}

f: A → B is defined as $f(x)=\left(\frac{x-2}{x-3}\right)$.

Let $x,\ y \in \text{A}$ such that $f(x)=f(y)$.

$$\Rightarrow \frac{x-2}{x-3}=\frac{y-2}{y-3}$$
$$\Rightarrow (x-2)(y-3)=(y-2)(x-3)$$
$$\Rightarrow xy-3x-2y+6=xy-3y-2x+6$$
$$\Rightarrow -3x-2y=-3y-2x$$
$$\Rightarrow 3x-2x=3y-2y$$
$$\Rightarrow x=y$$

∴ *f* is one-one.

Let *y* ∈B = **R** − {1}. Then, *y* ≠ 1.

The function *f* is onto if there exists *x* ∈A such that *f*(*x*) = *y.*

Now,

$$f(x)=y$$
$$\Rightarrow \frac{x-2}{x-3}=y$$
$$\Rightarrow x-2=xy-3y$$
$$\Rightarrow x(1-y)=-3y+2$$
$$\Rightarrow x=\frac{2-3y}{1-y}\in \text{A} \qquad [y\neq 1]$$

Thus, for any *y* ∈ B, there exists $\frac{2-3y}{1-y}\in \text{A}$ such that

$$f\left(\frac{2-3y}{1-y}\right)=\frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3}=\frac{2-3y-2+2y}{2-3y-3+3y}=\frac{-y}{-1}=y.$$

$\therefore f$ is onto.

Hence, function *f* is one-one and onto.

Question 11:

Let *f*: **R** → **R** be defined as $f(x) = x^4$. Choose the correct answer.

(A) *f* is one-one onto (B) *f* is many-one onto

(C) *f* is one-one but not onto (D) *f* is neither one-one nor onto

Answer

f: **R** → **R** is defined as $f(x)=x^4.$

Let *x*, *y* ∈ **R** such that *f*(*x*) = *f*(*y*).

$\Rightarrow x^4 = y^4$

$\Rightarrow x = \pm y$

∴ $f(x_1)=f(x_2)$ does not imply that $x_1 = x_2$.

For instance,

$f(1)=f(-1)=1$

∴ *f* is not one-one.

Consider an element 2 in co-domain **R**. It is clear that there does not exist any *x* in domain **R** such that *f*(*x*) = 2.

∴ *f* is not onto.

Hence, function *f* is neither one-one nor onto.

The correct answer is D.

Question 12:

Let *f*: **R** → **R** be defined as *f*(*x*) = 3*x*. Choose the correct answer.

(A) *f* is one-one onto (B) *f* is many-one onto

(C) *f* is one-one but not onto (D) *f* is neither one-one nor onto

Answer

f: **R** → **R** is defined as $f(x) = 3x$.

Let $x, y \in$ **R** such that $f(x) = f(y)$.

$\Rightarrow 3x = 3y$

$\Rightarrow x = y$

$\therefore f$ is one-one.

Also, for any real number (y) in co-domain **R**, there exists $\frac{y}{3}$ in **R** such that $f\left(\frac{y}{3}\right) = 3\left(\frac{y}{3}\right) = y$.

$\therefore f$ is onto.

Hence, function f is one-one and onto.

The correct answer is A.

Exercise 1.3

Question 1:

Let *f*: {1, 3, 4} → {1, 2, 5} and *g*: {1, 2, 5} → {1, 3} be given by *f* = {(1, 2), (3, 5), (4, 1)} and *g* = {(1, 3), (2, 3), (5, 1)}. Write down *gof*.

Answer

The functions *f*: {1, 3, 4} → {1, 2, 5} and *g*: {1, 2, 5} → {1, 3} are defined as *f* = {(1, 2), (3, 5), (4, 1)} and *g* = {(1, 3), (2, 3), (5, 1)}.

$gof(1) = g(f(1)) = g(2) = 3 \qquad [f(1) = 2 \text{ and } g(2) = 3]$

$gof(3) = g(f(3)) = g(5) = 1 \qquad [f(3) = 5 \text{ and } g(5) = 1]$

$gof(4) = g(f(4)) = g(1) = 3 \qquad [f(4) = 1 \text{ and } g(1) = 3]$

$\therefore gof = \{(1,3),(3,1),(4,3)\}$

Question 2:

Let *f*, *g* and *h* be functions from **R** to **R**. Show that

$(f+g)oh = foh + goh$

$(f \cdot g)oh = (foh)\cdot(goh)$

Answer

To prove:

$(f+g)oh = foh + goh$

Consider:

$((f+g)oh)(x)$

$= (f+g)(h(x))$

$= f(h(x)) + g(h(x))$

$= (foh)(x) + (goh)(x)$

$= \{(foh)+(goh)\}(x)$

$\therefore ((f+g)oh)(x) = \{(foh)+(goh)\}(x) \qquad \forall x \in \mathbf{R}$

Hence, $(f+g)oh = foh + goh$.

To prove:

$(f \cdot g)oh = (foh)\cdot(goh)$

Consider:

$((f \cdot g)oh)(x)$

$= (f \cdot g)(h(x))$

$= f(h(x)).g(h(x))$

$= (foh)(x).(goh)(x)$

$= \{(foh).(goh)\}(x)$

$\therefore ((f \cdot g)oh)(x) = \{(foh).(goh)\}(x)\ \forall x \in \mathrm{R}$

Hence, $(f \cdot g)oh = (foh)\cdot(goh)$.

Question 3:

Find *gof* and *fog*, if

(i) $f(x) = |x|$ and $g(x) = |5x-2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

Answer

(i) $f(x) = |x|$ and $g(x) = |5x-2|$

$\therefore (gof)(x) = g(f(x)) = g(|x|) = |5|x|-2|$

$(fog)(x) = f(g(x)) = f(|5x-2|) = ||5x-2|| = |5x-2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$

$\therefore (gof)(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = 2x$

$(fog)(x) = f(g(x)) = f\left(x^{\frac{1}{3}}\right) = 8\left(x^{\frac{1}{3}}\right)^3 = 8x$

Question 4:

If $f(x)=\frac{(4x+3)}{(6x-4)}, x\neq\frac{2}{3}$, show that f o $f(x) = x$, for all $x\neq\frac{2}{3}$. What is the inverse of f?

Answer

It is given that $f(x)=\frac{(4x+3)}{(6x-4)}, x\neq\frac{2}{3}$.

$$(fof)(x)=f(f(x))=f\left(\frac{4x+3}{6x-4}\right)$$

$$=\frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}=\frac{16x+12+18x-12}{24x+18-24x+16}=\frac{34x}{34}=x$$

$$\text{Therefore, } fof(x)=x, \text{ for all } x\neq\frac{2}{3}.$$

$$\Rightarrow fof=I$$

Hence, the given function f is invertible and the inverse of f is f itself.

Question 5:

State with reason whether following functions have inverse

(i) f: {1, 2, 3, 4} → {10} with

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

(ii) g: {5, 6, 7, 8} → {1, 2, 3, 4} with

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

(iii) h: {2, 3, 4, 5} → {7, 9, 11, 13} with

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

Answer

(i) f: {1, 2, 3, 4} → {10}defined as:

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

From the given definition of f, we can see that f is a many one function as: $f(1) = f(2) = f(3) = f(4) = 10$

∴f is not one-one.

Hence, function f does not have an inverse.

(ii) g: {5, 6, 7, 8} → {1, 2, 3, 4} defined as:

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

From the given definition of *g*, it is seen that *g* is a many one function as: *g*(5) = *g*(7) = 4.

∴*g* is not one-one,

Hence, function g does not have an inverse.

(iii) *h*: {2, 3, 4, 5} → {7, 9, 11, 13} defined as:

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

It is seen that all distinct elements of the set {2, 3, 4, 5} have distinct images under *h*.

∴Function *h* is one-one.

Also, *h* is onto since for every element *y* of the set {7, 9, 11, 13}, there exists an element *x* in the set {2, 3, 4, 5}such that *h*(*x*) = *y*.

Thus, *h* is a one-one and onto function. Hence, *h* has an inverse.

Question 6:

Show that *f*: [−1, 1] → **R**, given by $f(x)=\frac{x}{(x+2)}$ is one-one. Find the inverse of the function *f*: [−1, 1] → Range *f*.

(Hint: For *y* ∈Range *f*, *y* = $f(x)=\frac{x}{x+2}$, for some *x* in [−1, 1], i.e., $x=\frac{2y}{(1-y)}$)

Answer

f: [−1, 1] → R is given as $f(x)=\frac{x}{(x+2)}$.

Let *f*(*x*) = *f*(*y*).

$\Rightarrow \frac{x}{x+2}=\frac{y}{y+2}$

$\Rightarrow xy+2x=xy+2y$

$\Rightarrow 2x=2y$

$\Rightarrow x=y$

∴ *f* is a one-one function.

It is clear that *f*: [−1, 1] → Range *f* is onto.

∴ *f*: [−1, 1] → Range *f* is one-one and onto and therefore, the inverse of the function:

f: [−1, 1] → Range *f* exists.

Let *g*: Range *f* → [−1, 1] be the inverse of *f*.

Let *y* be an arbitrary element of range *f*.

Since *f*: [−1, 1] → Range *f* is onto, we have:

$$y = f(x) \text{ for same } x \in [-1,\ 1]$$

$$\Rightarrow y = \frac{x}{x+2}$$

$$\Rightarrow xy + 2y = x$$

$$\Rightarrow x(1-y) = 2y$$

$$\Rightarrow x = \frac{2y}{1-y},\ y \neq 1$$

Now, let us define *g*: Range *f* → [−1, 1] as

$$g(y) = \frac{2y}{1-y},\ y \neq 1.$$

$$\text{Now, } (gof)(x) = g(f(x)) = g\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1-\frac{x}{x+2}} = \frac{2x}{x+2-x} = \frac{2x}{2} = x$$

$$(fog)(y) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2} = \frac{2y}{2y+2-2y} = \frac{2y}{2} = y$$

∴*gof* = $I_{[-1,\ 1]}$ and *fog* = $I_{\text{Range } f}$

∴ $f^{-1} = g$

⇒ $f^{-1}(y) = \frac{2y}{1-y}, y \neq 1$

Question 7:

Consider *f*: **R** → **R** given by *f*(*x*) = 4*x* + 3. Show that *f* is invertible. Find the inverse of *f*.

Answer

f: **R** → **R** is given by,

f(*x*) = 4*x* + 3

One-one:

Let $f(x) = f(y)$.

$\Rightarrow 4x+3=4y+3$

$\Rightarrow 4x=4y$

$\Rightarrow x=y$

∴ f is a one-one function.

Onto:

For $y \in \mathbf{R}$, let $y = 4x + 3$.

$\Rightarrow x=\dfrac{y-3}{4}\in \mathbf{R}$

Therefore, for any $y \in \mathbf{R}$, there exists $x=\dfrac{y-3}{4}\in \mathbf{R}$ such that

$f(x)=f\left(\dfrac{y-3}{4}\right)=4\left(\dfrac{y-3}{4}\right)+3=y.$

∴ f is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define g: $\mathbf{R}\rightarrow \mathbf{R}$ by $g(x)=\dfrac{y-3}{4}$.

$\text{Now,}(gof)(x)=g(f(x))=g(4x+3)=\dfrac{(4x+3)-3}{4}=x$

$(fog)(y)=f(g(y))=f\left(\dfrac{y-3}{4}\right)=4\left(\dfrac{y-3}{4}\right)+3=y-3+3=y$

∴ $gof=fog=I_R$

Hence, f is invertible and the inverse of f is given by

$f^{-1}(y)=g(y)=\dfrac{y-3}{4}.$

Question 8:

Consider f: $\mathbf{R}_+ \rightarrow [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of given f by $f^{-1}(y)=\sqrt{y-4}$, where $\mathbf{R}_+$ is the set of all non-negative real numbers.

Answer

f: $\mathbf{R}_+ \rightarrow [4, \infty)$ is given as $f(x) = x^2 + 4$.

One-one:

Let $f(x) = f(y)$.

$$\Rightarrow x^2 + 4 = y^2 + 4$$
$$\Rightarrow x^2 = y^2$$
$$\Rightarrow x = y \qquad [\text{as } x = y \in \mathbf{R}_+]$$

∴ f is a one-one function.

Onto:

For $y \in [4, \infty)$, let $y = x^2 + 4$.

$$\Rightarrow x^2 = y - 4 \geq 0 \qquad [\text{as } y \geq 4]$$
$$\Rightarrow x = \sqrt{y-4} \geq 0$$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \sqrt{y-4} \in \mathbf{R}$ such that

$$f(x) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = y - 4 + 4 = y.$$

∴ f is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define g: $[4, \infty) \to \mathbf{R}_+$ by,

$$g(y) = \sqrt{y-4}$$

$$\text{Now, } gof(x) = g(f(x)) = g(x^2 + 4) = \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$$

$$\text{And, } fog(y) = f(g(y)) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = (y-4) + 4 = y$$

$$\therefore gof = fo\,g = I_{\mathbf{R}+}$$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \sqrt{y-4}.$$

Question 21:

Find the values of $\tan^{-1}\sqrt{3} - \cot^{-1}\left(-\sqrt{3}\right)$ is equal to

(A) π **(B)** $-\frac{\pi}{2}$ **(C)** 0 **(D)** $2\sqrt{3}$

Answer

Let $\tan^{-1}\sqrt{3} = x$. Then, $\tan x = \sqrt{3} = \tan\frac{\pi}{3}$ where $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Let $\cot^{-1}\left(-\sqrt{3}\right) = y$.

Then, $\cot y = -\sqrt{3} = -\cot\left(\frac{\pi}{6}\right) = \cot\left(\pi - \frac{\pi}{6}\right) = \cot\frac{5\pi}{6}$ where $\frac{5\pi}{6} \in (0, \pi)$.

The range of the principal value branch of $\cot^{-1}$ is $(0, \pi)$.

$\therefore \cot^{-1}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$

$\therefore \tan^{-1}\sqrt{3} - \cot^{-1}\left(-\sqrt{3}\right) = \frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi - 5\pi}{6} = \frac{-3\pi}{6} = -\frac{\pi}{2}$

The correct answer is B.

Question 9:

Consider f: $\mathbf{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with

$f^{-1}(y) = \left(\frac{\left(\sqrt{y+6}\right)-1}{3}\right)$.

Answer

f: $\mathbf{R}_+ \to [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be an arbitrary element of $[-5, \infty)$.

Let $y = 9x^2 + 6x - 5$.

$\Rightarrow y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$

$\Rightarrow (3x+1)^2 = y + 6$

$\Rightarrow 3x + 1 = \sqrt{y+6} \quad [\text{as } y \geq -5 \Rightarrow y + 6 > 0]$

$\Rightarrow x = \frac{\sqrt{y+6} - 1}{3}$

$\therefore f$ is onto, thereby range $f = [-5, \infty)$.

Let us define g: $[-5, \infty) \rightarrow \mathbf{R}_+$ as $g(y) = \dfrac{\sqrt{y+6}-1}{3}$.

We now have:

$$\begin{aligned}(gof)(x) = g(f(x)) = g(9x^2+6x-5) &= g\left((3x+1)^2-6\right) \\ &= \frac{\sqrt{(3x+1)^2-6+6}-1}{3} \\ &= \frac{3x+1-1}{3} = x\end{aligned}$$

$$\begin{aligned}\text{And, }(fog)(y) = f(g(y)) = f\left(\frac{\sqrt{y+6}-1}{3}\right) &= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right)+1\right]^2-6 \\ &= \left(\sqrt{y+6}\right)^2-6 = y+6-6 = y\end{aligned}$$

$\therefore gof = I_{\mathbf{R}_+}$ and $fog = I_{[-5,\infty)}$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}.$$

Question 10:

Let f: $X \rightarrow Y$ be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$,

$fog_1(y) = I_Y(y) = fog_2(y)$. Use one-one ness of f).

Answer

Let f: $X \rightarrow Y$ be an invertible function.

Also, suppose f has two inverses (say g_1 and g_2).

Then, for all $y \in Y$, we have:

$$fog_1(y) = I_Y(y) = fog_2(y)$$
$$\Rightarrow f(g_1(y)) = f(g_2(y))$$
$$\Rightarrow g_1(y) = g_2(y) \qquad [f \text{ is invertible} \Rightarrow f \text{ is one-one}]$$
$$\Rightarrow g_1 = g_2 \qquad [g \text{ is one-one}]$$

Hence, *f* has a unique inverse.

Question 11:

Consider *f*: {1, 2, 3} → {*a*, *b*, *c*} given by *f*(1) = *a*, *f*(2) = *b* and *f*(3) = *c*. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Answer

Function *f*: {1, 2, 3} → {*a*, *b*, *c*} is given by,

f(1) = *a*, *f*(2) = *b,* and *f*(3) = *c*

If we define *g*: {*a*, *b*, *c*} → {1, 2, 3} as *g*(*a*) = 1, *g*(*b*) = 2, *g*(*c*) = 3, then we have:

$$(fog)(a) = f(g(a)) = f(1) = a$$
$$(fog)(b) = f(g(b)) = f(2) = b$$
$$(fog)(c) = f(g(c)) = f(3) = c$$

And,

$$(gof)(1) = g(f(1)) = g(a) = 1$$
$$(gof)(2) = g(f(2)) = g(b) = 2$$
$$(gof)(3) = g(f(3)) = g(c) = 3$$

∴ $gof = I_X$ and $fog = I_Y$, where *X* = {1, 2, 3} and *Y*= {*a*, *b*, *c*}.

Thus, the inverse of *f* exists and $f^{-1} = g$.

∴f^{-1}: {*a*, *b*, *c*} → {1, 2, 3} is given by,

$f^{-1}(a) = 1$, $f^{-1}(b) = 2$, $f^{-1}(c) = 3$

Let us now find the inverse of f^{-1} i.e., find the inverse of *g*.

If we define *h*: {1, 2, 3} → {*a*, *b*, *c*} as

h(1) = *a*, *h*(2) = *b*, *h*(3) = *c*, then we have:

$(goh)(1) = g(h(1)) = g(a) = 1$

$(goh)(2) = g(h(2)) = g(b) = 2$

$(goh)(3) = g(h(3)) = g(c) = 3$

And,

$(hog)(a) = h(g(a)) = h(1) = a$

$(hog)(b) = h(g(b)) = h(2) = b$

$(hog)(c) = h(g(c)) = h(3) = c$

$\therefore goh = I_X$ and $hog = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

It can be noted that $h = f$.

Hence, $(f^{-1})^{-1} = f$.

Question 12:

Let $f: X \to Y$ be an invertible function. Show that the inverse of f^{-1} is f, i.e.,
$(f^{-1})^{-1} = f$.

Answer

Let $f: X \to Y$ be an invertible function.

Then, there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$.

Here, $f^{-1} = g$.

Now, $gof = I_X$ and $fog = I_Y$

$\Rightarrow f^{-1}of = I_X$ and $fof^{-1} = I_Y$

Hence, $f^{-1}: Y \to X$ is invertible and f is the inverse of f^{-1}

i.e., $(f^{-1})^{-1} = f$.

Question 13:

If f: **R** → **R** be given by $f(x) = (3 - x^3)^{\frac{1}{3}}$, then $fof(x)$ is

(A) $\frac{1}{x^3}$ (B) x^3 (C) x (D) $(3 - x^3)$

Answer

f: **R** → **R** is given as $f(x)=\left(3-x^3\right)^{\frac{1}{3}}$.

$$f(x)=\left(3-x^3\right)^{\frac{1}{3}}$$

$$\therefore fof(x)=f\left(f(x)\right)=f\left(\left(3-x^3\right)^{\frac{1}{3}}\right)=\left[3-\left(\left(3-x^3\right)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$=\left[3-\left(3-x^3\right)\right]^{\frac{1}{3}}=\left(x^3\right)^{\frac{1}{3}}=x$$

$$\therefore fof(x)=x$$

The correct answer is C.

Question 14:

Let $f:\mathbf{R}-\left\{-\frac{4}{3}\right\}\to\mathbf{R}$ be a function defined as $f(x)=\frac{4x}{3x+4}$. The inverse of *f* is map *g*: Range $f\to\mathbf{R}-\left\{-\frac{4}{3}\right\}$ given by

(A) $g(y)=\frac{3y}{3-4y}$ (B) $g(y)=\frac{4y}{4-3y}$

(C) $g(y)=\frac{4y}{3-4y}$ (D) $g(y)=\frac{3y}{4-3y}$

Answer

It is given that $f:\mathbf{R}-\left\{-\frac{4}{3}\right\}\to\mathbf{R}$ is defined as $f(x)=\frac{4x}{3x+4}$.

Let *y* be an arbitrary element of Range *f*.

Then, there exists *x* ∈ $\mathbf{R}-\left\{-\frac{4}{3}\right\}$ such that $y=f(x)$.

$$\Rightarrow y=\frac{4x}{3x+4}$$
$$\Rightarrow 3xy+4y=4x$$
$$\Rightarrow x(4-3y)=4y$$
$$\Rightarrow x=\frac{4y}{4-3y}$$

Let us define *g*: Range $f\to\mathbf{R}-\left\{-\frac{4}{3}\right\}$ as $g(y)=\frac{4y}{4-3y}$.

Now, $(g \circ f)(x) = g(f(x)) = g\left(\frac{4x}{3x+4}\right)$

$$= \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)} = \frac{16x}{12x+16-12x} = \frac{16x}{16} = x$$

$$\text{And,} (f \circ g)(y) = f(g(y)) = f\left(\frac{4y}{4-3y}\right)$$

$$= \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4} = \frac{16y}{12y+16-12y} = \frac{16y}{16} = y$$

∴ $gof = I_{R-\left\{-\frac{4}{3}\right\}}$ and $fog = I_{\text{Range} f}$

Thus, *g* is the inverse of *f* i.e., $f^{-1} = g$.

Hence, the inverse of *f* is the map *g*: Range $f \to \mathbf{R} - \left\{-\frac{4}{3}\right\}$, which is given by

$$g(y) = \frac{4y}{4-3y}.$$

The correct answer is B.

Exercise 1.4

Question 1:

Determine whether or not each of the definition of given below gives a binary operation. In the event that * is not a binary operation, give justification for this.

(i) On $\mathbf{Z}^+$, define * by $a * b = a - b$

(ii) On $\mathbf{Z}^+$, define * by $a * b = ab$

(iii) On **R**, define * by $a * b = ab^2$

(iv) On $\mathbf{Z}^+$, define * by $a * b = |a - b|$

(v) On $\mathbf{Z}^+$, define * by $a * b = a$

Answer

(i) On $\mathbf{Z}^+$, * is defined by $a * b = a - b$.

It is not a binary operation as the image of (1, 2) under * is $1 * 2 = 1 - 2 = -1 \notin \mathbf{Z}^+$.

(ii) On $\mathbf{Z}^+$, * is defined by $a * b = ab$.

It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element ab in $\mathbf{Z}^+$.

This means that * carries each pair (a, b) to a unique element $a * b = ab$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

(iii) On **R**, * is defined by $a * b = ab^2$.

It is seen that for each $a, b \in \mathbf{R}$, there is a unique element ab^2 in **R**.

This means that * carries each pair (a, b) to a unique element $a * b = ab^2$ in **R**.

Therefore, * is a binary operation.

(iv) On $\mathbf{Z}^+$, * is defined by $a * b = |a - b|$.

It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element $|a - b|$ in $\mathbf{Z}^+$.

This means that * carries each pair (a, b) to a unique element $a * b = |a - b|$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

(v) On $\mathbf{Z}^+$, * is defined by $a * b = a$.

* carries each pair (a, b) to a unique element $a * b = a$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

Question 2:

For each binary operation * defined below, determine whether * is commutative or associative.

(i) On **Z**, define $a * b = a - b$

(ii) On **Q**, define $a * b = ab + 1$

(iii) On **Q**, define $a * b = \frac{ab}{2}$

(iv) On $\mathbf{Z}^+$, define $a * b = 2^{ab}$

(v) On $\mathbf{Z}^+$, define $a * b = a^b$

(vi) On **R** − {−1}, define $a * b = \frac{a}{b+1}$

Answer

(i) On **Z**, * is defined by $a * b = a - b$.

It can be observed that 1 * 2 = 1 − 2 = 1 and 2 * 1 = 2 − 1 = 1.

∴1 * 2 ≠ 2 * 1; where 1, 2 ∈ **Z**

Hence, the operation * is not commutative.

Also we have:

(1 * 2) * 3 = (1 − 2) * 3 = −1 * 3 = −1 − 3 = −4

1 * (2 * 3) = 1 * (2 − 3) = 1 * −1 = 1 − (−1) = 2

∴(1 * 2) * 3 ≠ 1 * (2 * 3) ; where 1, 2, 3 ∈ **Z**

Hence, the operation * is not associative.

(ii) On **Q**, * is defined by $a * b = ab + 1$.

It is known that:

ab = *ba* ☐ *a, b* ∈ **Q**

⇒ *ab* + 1 = *ba* + 1 ☐ *a, b* ∈ **Q**

⇒ *a* * *b* = *a* * *b* ☐ *a, b* ∈ **Q**

Therefore, the operation * is commutative.

It can be observed that:

(1 * 2) * 3 = (1 × 2 + 1) * 3 = 3 * 3 = 3 × 3 + 1 = 10

1 * (2 * 3) = 1 * (2 × 3 + 1) = 1 * 7 = 1 × 7 + 1 = 8

∴(1 * 2) * 3 ≠ 1 * (2 * 3) ; where 1, 2, 3 ∈ **Q**

Therefore, the operation * is not associative.