

Exercise 2.1

Question 1:

Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$

Answer

Let $\sin^{-1}\left(-\frac{1}{2}\right) = y$. Then $\sin y = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of \sin^{-1} is

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$.

Therefore, the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}$.

Question 2:

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Answer

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y$. Then, $\cos y = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of \cos^{-1} is

$[0, \pi]$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

Therefore, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

Question 3:

Find the principal value of $\operatorname{cosec}^{-1}(2)$

Answer

Let $\operatorname{cosec}^{-1}(2) = y$. Then, $\operatorname{cosec} y = 2 = \operatorname{cosec}\left(\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}(2)$ is $\frac{\pi}{6}$.

Question 4:

Find the principal value of $\tan^{-1}(-\sqrt{3})$

Answer

Let $\tan^{-1}(-\sqrt{3}) = y$. Then, $\tan y = -\sqrt{3} = -\tan \frac{\pi}{3} = \tan\left(-\frac{\pi}{3}\right)$.

We know that the range of the principal value branch of \tan^{-1} is

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{3}\right)$ is $-\sqrt{3}$.

Therefore, the principal value of $\tan^{-1}(\sqrt{3})$ is $-\frac{\pi}{3}$.

Question 5:

Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$

Answer

Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$. Then, $\cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$.

We know that the range of the principal value branch of \cos^{-1} is

$[0, \pi]$ and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$.

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $\frac{2\pi}{3}$.

Question 6:

Find the principal value of $\tan^{-1}(-1)$

Answer

$$\tan y = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right).$$

Let $\tan^{-1}(-1) = y$. Then,

We know that the range of the principal value branch of \tan^{-1} is

$$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \text{ and } \tan\left(-\frac{\pi}{4}\right) = -1.$$

Therefore, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}$.

Question 7:

Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Answer

$$\text{Let } \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y. \text{ Then, } \sec y = \frac{2}{\sqrt{3}} = \sec\left(\frac{\pi}{6}\right).$$

We know that the range of the principal value branch of \sec^{-1} is

$$[0, \pi] - \left\{\frac{\pi}{2}\right\} \text{ and } \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}.$$

Therefore, the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}$.

Question 8:

Find the principal value of $\cot^{-1}(\sqrt{3})$

Answer

$$\text{Let } \cot^{-1}(\sqrt{3}) = y. \text{ Then, } \cot y = \sqrt{3} = \cot\left(\frac{\pi}{6}\right).$$

We know that the range of the principal value branch of \cot^{-1} is $(0, \pi)$ and

$$\cot\left(\frac{\pi}{6}\right) = \sqrt{3}.$$

Therefore, the principal value of $\cot^{-1}(\sqrt{3})$ is $\frac{\pi}{6}$.

Question 9:

Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Answer

$$\text{Let } \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y. \text{ Then, } \cos y = -\frac{1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right).$$

We know that the range of the principal value branch of \cos^{-1} is $[0, \pi]$ and

$$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

Question 10:

Find the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$

Answer

$$\text{Let } \operatorname{cosec}^{-1}(-\sqrt{2}) = y. \text{ Then, } \operatorname{cosec} y = -\sqrt{2} = -\operatorname{cosec}\left(\frac{\pi}{4}\right) = \operatorname{cosec}\left(-\frac{\pi}{4}\right).$$

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \text{ and } \operatorname{cosec}\left(-\frac{\pi}{4}\right) = -\sqrt{2}.$$

Therefore, the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$ is $-\frac{\pi}{4}$.

Question 11:

$$\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$$

Find the value of

Answer

Let $\tan^{-1}(1) = x$. Then, $\tan x = 1 = \tan \frac{\pi}{4}$.

$$\therefore \tan^{-1}(1) = \frac{\pi}{4}$$

Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$. Then, $\cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$.

$$\therefore \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = z$. Then, $\sin z = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$.

$$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\therefore \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$$

$$= \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$$

Question 12:

$$\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$$

Find the value of

Answer

$$\text{Let } \cos^{-1}\left(\frac{1}{2}\right) = x. \text{ Then, } \cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right).$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\text{Let } \sin^{-1}\left(\frac{1}{2}\right) = y. \text{ Then, } \sin y = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right).$$

$$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + \frac{2\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

Question 13:

Find the value of y if $\sin^{-1} x = y$, then

(A) $0 \leq y \leq \pi$ (B) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Answer

It is given that $\sin^{-1} x = y$.

We know that the range of the principal value branch of \sin^{-1} is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Therefore, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Question 14:

Find the value of $\tan^{-1} \sqrt{3} - \sec^{-1}(-2)$ is equal to

(A) π (B) $-\frac{\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

Answer

Let $\tan^{-1}\sqrt{3} = x$. Then, $\tan x = \sqrt{3} = \tan \frac{\pi}{3}$.

We know that the range of the principal value branch of \tan^{-1} is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

Let $\sec^{-1}(-2) = y$. Then, $\sec y = -2 = -\sec\left(\frac{\pi}{3}\right) = \sec\left(\pi - \frac{\pi}{3}\right) = \sec \frac{2\pi}{3}$.

We know that the range of the principal value branch of \sec^{-1} is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$$\therefore \sec^{-1}(-2) = \frac{2\pi}{3}$$

$$\text{Hence, } \tan^{-1}(\sqrt{3}) - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$$

Exercise 2.2

Question 1:

$$3 \sin^{-1} x = \sin^{-1} (3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2} \right]$$

Prove

Answer

$$\text{To prove: } 3 \sin^{-1} x = \sin^{-1} (3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2} \right]$$

Let $x = \sin \theta$. Then, $\sin^{-1} x = \theta$.

We have,

$$\begin{aligned} \text{R.H.S.} &= \sin^{-1} (3x - 4x^3) = \sin^{-1} (3 \sin \theta - 4 \sin^3 \theta) \\ &= \sin^{-1} (\sin 3\theta) \\ &= 3\theta \\ &= 3 \sin^{-1} x \\ &= \text{L.H.S.} \end{aligned}$$

Question 2:

$$3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x), x \in \left[\frac{1}{2}, 1 \right]$$

Prove

Answer

$$\text{To prove: } 3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x), x \in \left[\frac{1}{2}, 1 \right]$$

Let $x = \cos \theta$. Then, $\cos^{-1} x = \theta$.

We have,

$$\begin{aligned}
 \text{R.H.S.} &= \cos^{-1}(4x^3 - 3x) \\
 &= \cos^{-1}(4\cos^3\theta - 3\cos\theta) \\
 &= \cos^{-1}(\cos 3\theta) \\
 &= 3\theta \\
 &= 3\cos^{-1}x \\
 &= \text{L.H.S.}
 \end{aligned}$$

Question 3:

Prove $\tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} = \tan^{-1} \frac{1}{2}$

Answer

To prove: $\tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} = \tan^{-1} \frac{1}{2}$

$$\begin{aligned}
 \text{L.H.S.} &= \tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} \\
 &= \tan^{-1} \frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11} \cdot \frac{7}{24}} \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\
 &= \tan^{-1} \frac{48+77}{11 \times 24 - 14} \\
 &= \tan^{-1} \frac{125}{264-14} = \tan^{-1} \frac{125}{250} = \tan^{-1} \frac{1}{2} = \text{R.H.S.}
 \end{aligned}$$

Question 4:

Prove $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$

Answer

To prove: $2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$

$$\begin{aligned}
 \text{L.H.S.} &= 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{2 \cdot \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} + \tan^{-1} \frac{1}{7} \quad \left[2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \right] \\
 &= \tan^{-1} \frac{1}{\left(\frac{3}{4}\right)} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{4}{3} + \tan^{-1} \frac{1}{7} \\
 &= \tan^{-1} \frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \cdot \frac{1}{7}} \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\
 &= \tan^{-1} \frac{\left(\frac{28+3}{21}\right)}{\left(\frac{21-4}{21}\right)} \\
 &= \tan^{-1} \frac{31}{17} = \text{R.H.S.}
 \end{aligned}$$

Question 5:

Write the function in the simplest form:

$$\tan^{-1} \frac{\sqrt{1+x^2} - 1}{x}, \quad x \neq 0$$

Answer

$$\begin{aligned} & \tan^{-1} \frac{\sqrt{1+x^2}-1}{x} \\ \text{Put } x &= \tan \theta \Rightarrow \theta = \tan^{-1} x \\ \therefore \tan^{-1} \frac{\sqrt{1+x^2}-1}{x} &= \tan^{-1} \left(\frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta} \right) \\ &= \tan^{-1} \left(\frac{\sec \theta - 1}{\tan \theta} \right) = \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right) \\ &= \tan^{-1} \left(\frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right) \\ &= \tan^{-1} \left(\tan \frac{\theta}{2} \right) = \frac{\theta}{2} = \frac{1}{2} \tan^{-1} x \end{aligned}$$

Question 6:

Write the function in the simplest form:

$$\tan^{-1} \frac{1}{\sqrt{x^2-1}}, \quad |x| > 1$$

Answer

$$\tan^{-1} \frac{1}{\sqrt{x^2-1}}, \quad |x| > 1$$

Put $x = \operatorname{cosec} \theta \Rightarrow \theta = \operatorname{cosec}^{-1} x$

$$\begin{aligned} \therefore \tan^{-1} \frac{1}{\sqrt{x^2-1}} &= \tan^{-1} \frac{1}{\sqrt{\operatorname{cosec}^2 \theta - 1}} \\ &= \tan^{-1} \left(\frac{1}{\cot \theta} \right) = \tan^{-1} (\tan \theta) \\ &= \theta = \operatorname{cosec}^{-1} x = \frac{\pi}{2} - \sec^{-1} x \quad \left[\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2} \right] \end{aligned}$$

Question 7:

Write the function in the simplest form:

$$\tan^{-1} \left(\sqrt{\frac{1 - \cos x}{1 + \cos x}} \right), x < \pi$$

Answer

$$\tan^{-1} \left(\sqrt{\frac{1 - \cos x}{1 + \cos x}} \right), x < \pi$$

$$\tan^{-1} \left(\sqrt{\frac{1 - \cos x}{1 + \cos x}} \right) = \tan^{-1} \left(\sqrt{\frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}}} \right)$$

$$= \tan^{-1} \left(\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} \right) = \tan^{-1} \left(\tan \frac{x}{2} \right)$$

$$= \frac{x}{2}$$

Question 8:

Write the function in the simplest form:

$$\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right), 0 < x < \pi$$

Answer

$$\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right)$$

$$= \tan^{-1} \left(\frac{1 - \frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}} \right)$$

$$= \tan^{-1} \left(\frac{1 - \tan x}{1 + \tan x} \right)$$

$$= \tan^{-1}(1) - \tan^{-1}(\tan x)$$

$$\left[\tan^{-1} \frac{x - y}{1 - xy} = \tan^{-1} x - \tan^{-1} y \right]$$

$$= \frac{\pi}{4} - x$$

Question 9:

Write the function in the simplest form:

$$\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}, \quad |x| < a$$

Answer

$$\tan^{-1} \frac{x}{\sqrt{a^2 - x^2}}$$

$$\text{Put } x = a \sin \theta \Rightarrow \frac{x}{a} = \sin \theta \Rightarrow \theta = \sin^{-1} \left(\frac{x}{a} \right)$$

$$\therefore \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} = \tan^{-1} \left(\frac{a \sin \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} \right)$$

$$= \tan^{-1} \left(\frac{a \sin \theta}{a \sqrt{1 - \sin^2 \theta}} \right) = \tan^{-1} \left(\frac{a \sin \theta}{a \cos \theta} \right)$$

$$= \tan^{-1} (\tan \theta) = \theta = \sin^{-1} \frac{x}{a}$$

Question 10:

Write the function in the simplest form:

$$\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right), \quad a > 0; \quad \frac{-a}{\sqrt{3}} \leq x \leq \frac{a}{\sqrt{3}}$$

Answer

$$\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right)$$

$$\text{Put } x = a \tan \theta \Rightarrow \frac{x}{a} = \tan \theta \Rightarrow \theta = \tan^{-1} \frac{x}{a}$$

$$\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right) = \tan^{-1}\left(\frac{3a^2 \cdot a \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a \cdot a^2 \tan^2 \theta}\right)$$

$$= \tan^{-1}\left(\frac{3a^3 \tan \theta - a^3 \tan^3 \theta}{a^3 - 3a^3 \tan^2 \theta}\right)$$

$$= \tan^{-1}\left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta$$

$$= 3 \tan^{-1} \frac{x}{a}$$

Question 11:

$$\tan^{-1}\left[2 \cos\left(2 \sin^{-1} \frac{1}{2}\right)\right]$$

Find the value of

Answer

$$\text{Let } \sin^{-1} \frac{1}{2} = x. \quad \text{Then, } \sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right).$$

$$\therefore \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

$$\therefore \tan^{-1}\left[2 \cos\left(2 \sin^{-1} \frac{1}{2}\right)\right] = \tan^{-1}\left[2 \cos\left(2 \times \frac{\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[2 \cos \frac{\pi}{3}\right] = \tan^{-1}\left[2 \times \frac{1}{2}\right]$$

$$= \tan^{-1} 1 = \frac{\pi}{4}$$

Question 12:

Find the value of $\cot(\tan^{-1} a + \cot^{-1} a)$

Answer

$$\begin{aligned} & \cot(\tan^{-1} a + \cot^{-1} a) \\ &= \cot\left(\frac{\pi}{2}\right) \quad \left[\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right] \\ &= 0 \end{aligned}$$

Question 13:

Find the value of $\tan \frac{1}{2} \left[\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right]$, $|x| < 1$, $y > 0$ and $xy < 1$

Answer

Let $x = \tan \theta$. Then, $\theta = \tan^{-1} x$.

$$\therefore \sin^{-1} \frac{2x}{1+x^2} = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) = \sin^{-1} (\sin 2\theta) = 2\theta = 2 \tan^{-1} x$$

Let $y = \tan \phi$. Then, $\phi = \tan^{-1} y$.

$$\therefore \cos^{-1} \frac{1-y^2}{1+y^2} = \cos^{-1} \left(\frac{1 - \tan^2 \phi}{1 + \tan^2 \phi} \right) = \cos^{-1} (\cos 2\phi) = 2\phi = 2 \tan^{-1} y$$

$$\therefore \tan \frac{1}{2} \left[\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right]$$

$$= \tan \frac{1}{2} [2 \tan^{-1} x + 2 \tan^{-1} y]$$

$$= \tan [\tan^{-1} x + \tan^{-1} y]$$

$$= \tan \left[\tan^{-1} \left(\frac{x+y}{1-xy} \right) \right]$$

$$= \frac{x+y}{1-xy}$$

Question 14:

If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x .

Answer

$$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$$

$$\Rightarrow \sin\left(\sin^{-1}\frac{1}{5}\right)\cos\left(\cos^{-1}x\right) + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right) = 1$$

$$\left[\sin(A+B) = \sin A \cos B + \cos A \sin B\right]$$

$$\Rightarrow \frac{1}{5} \times x + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right) = 1$$

$$\Rightarrow \frac{x}{5} + \cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right) = 1 \quad \dots(1)$$

Now, let $\sin^{-1}\frac{1}{5} = y$.

$$\text{Then, } \sin y = \frac{1}{5} \Rightarrow \cos y = \sqrt{1 - \left(\frac{1}{5}\right)^2} = \frac{2\sqrt{6}}{5} \Rightarrow y = \cos^{-1}\left(\frac{2\sqrt{6}}{5}\right).$$

$$\therefore \sin^{-1}\frac{1}{5} = \cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \quad \dots(2)$$

Let $\cos^{-1}x = z$.

$$\text{Then, } \cos z = x \Rightarrow \sin z = \sqrt{1 - x^2} \Rightarrow z = \sin^{-1}\left(\sqrt{1 - x^2}\right).$$

$$\therefore \cos^{-1}x = \sin^{-1}\left(\sqrt{1 - x^2}\right) \quad \dots(3)$$

From (1), (2), and (3) we have:

$$\frac{x}{5} + \cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right) \cdot \sin\left(\sin^{-1}\sqrt{1 - x^2}\right) = 1$$

$$\Rightarrow \frac{x}{5} + \frac{2\sqrt{6}}{5} \cdot \sqrt{1 - x^2} = 1$$

$$\Rightarrow x + 2\sqrt{6}\sqrt{1 - x^2} = 5$$

$$\Rightarrow 2\sqrt{6}\sqrt{1 - x^2} = 5 - x$$

On squaring both sides, we get:

$$(4)(6)(1-x^2) = 25 + x^2 - 10x$$

$$\Rightarrow 24 - 24x^2 = 25 + x^2 - 10x$$

$$\Rightarrow 25x^2 - 10x + 1 = 0$$

$$\Rightarrow (5x-1)^2 = 0$$

$$\Rightarrow (5x-1) = 0$$

$$\Rightarrow x = \frac{1}{5}$$

Hence, the value of x is $\frac{1}{5}$.

Question 15:

If $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of x .

Answer

$$\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)} \right] = \frac{\pi}{4}$$

$$\left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right]$$

$$\Rightarrow \tan^{-1} \left[\frac{(x-1)(x+2) + (x+1)(x-2)}{(x+2)(x-2) - (x-1)(x+1)} \right] = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left[\frac{x^2 + x - 2 + x^2 - x - 2}{x^2 - 4 - x^2 + 1} \right] = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left[\frac{2x^2 - 4}{-3} \right] = \frac{\pi}{4}$$

$$\Rightarrow \tan \left[\tan^{-1} \frac{4 - 2x^2}{3} \right] = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{4 - 2x^2}{3} = 1$$

$$\Rightarrow 4 - 2x^2 = 3$$

$$\Rightarrow 2x^2 = 4 - 3 = 1$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

Hence, the value of x is $\pm \frac{1}{\sqrt{2}}$.

Question 16:

Find the values of $\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$

Answer

$$\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$$

We know that $\sin^{-1}(\sin x) = x$ if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1}x$.

Here, $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now, $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$ can be written as:

$$\sin^{-1}\left(\sin \frac{2\pi}{3}\right) = \sin^{-1}\left[\sin\left(\pi - \frac{2\pi}{3}\right)\right] = \sin^{-1}\left(\sin \frac{\pi}{3}\right) \text{ where } \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(\sin \frac{2\pi}{3}\right) = \sin^{-1}\left(\sin \frac{\pi}{3}\right) = \frac{\pi}{3}$$

Question 17:

Find the values of $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$

Answer

$$\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$$

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{3\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan \frac{3\pi}{4}\right)$ can be written as:

$$\begin{aligned} \tan^{-1}\left(\tan \frac{3\pi}{4}\right) &= \tan^{-1}\left[-\tan\left(\frac{-3\pi}{4}\right)\right] = \tan^{-1}\left[-\tan\left(\pi - \frac{\pi}{4}\right)\right] \\ &= \tan^{-1}\left[-\tan \frac{\pi}{4}\right] = \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \text{ where } -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{aligned}$$

$$\therefore \tan^{-1}\left(\tan \frac{3\pi}{4}\right) = \tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] = \frac{-\pi}{4}$$

Question 18:

$$\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$$

Find the values of

Answer

$$\text{Let } \sin^{-1}\frac{3}{5} = x \quad \text{Then, } \sin x = \frac{3}{5} \Rightarrow \cos x = \sqrt{1 - \sin^2 x} = \frac{4}{5} \Rightarrow \sec x = \frac{5}{4}$$

$$\therefore \tan x = \sqrt{\sec^2 x - 1} = \sqrt{\frac{25}{16} - 1} = \frac{3}{4}$$

$$\therefore x = \tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \quad \dots(i)$$

$$\text{Now, } \cot^{-1}\frac{3}{2} = \tan^{-1}\frac{2}{3} \quad \dots(ii) \quad \left[\tan^{-1}\frac{1}{x} = \cot^{-1}x\right]$$

$$\text{Hence, } \tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$$

$$= \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right) \quad [\text{Using (i) and (ii)}]$$

$$= \tan\left(\tan^{-1}\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \cdot \frac{2}{3}}\right) \quad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$$

$$= \tan\left(\tan^{-1}\frac{17}{6}\right) = \frac{17}{6}$$

Question 19:

Find the values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

- (A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{7\pi}{6} \notin x \in [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{-7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right] \quad [\cos(2\pi + x) = \cos x]$$

$$= \cos^{-1}\left[\cos\frac{5\pi}{6}\right] \text{ where } \frac{5\pi}{6} \in [0, \pi]$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$$

The correct answer is B.

Question 20:

Find the values of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

- (A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

Answer

Let $\sin^{-1}\left(-\frac{1}{2}\right) = x$. Then, $\sin x = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(-\frac{\pi}{6}\right)$.

We know that the range of the principal value branch of \sin^{-1} is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$$

$$\therefore \sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

The correct answer is D.

Miscellaneous Solutions

Question 1:

Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{13\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right] = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right], \text{ where } \frac{\pi}{6} \in [0, \pi].$$

$$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right] = \frac{\pi}{6}$$

Question 2:

Find the value of $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Answer

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\frac{7\pi}{6} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ can be written as:

$$\begin{aligned} \tan^{-1}\left(\tan\frac{7\pi}{6}\right) &= \tan^{-1}\left[\tan\left(2\pi - \frac{5\pi}{6}\right)\right] && [\tan(2\pi - x) = -\tan x] \\ &= \tan^{-1}\left[-\tan\left(\frac{5\pi}{6}\right)\right] = \tan^{-1}\left[\tan\left(-\frac{5\pi}{6}\right)\right] = \tan^{-1}\left[\tan\left(\pi - \frac{5\pi}{6}\right)\right] \\ &= \tan^{-1}\left[\tan\left(\frac{\pi}{6}\right)\right], \text{ where } \frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \therefore \tan^{-1}\left(\tan\frac{7\pi}{6}\right) &= \tan^{-1}\left(\tan\frac{\pi}{6}\right) = \frac{\pi}{6} \end{aligned}$$

Question 3:

Prove $2 \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{24}{7}$

Answer

Let $\sin^{-1} \frac{3}{5} = x$. Then, $\sin x = \frac{3}{5}$.

$$\Rightarrow \cos x = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

$$\therefore \tan x = \frac{3}{4}$$

$$\therefore x = \tan^{-1} \frac{3}{4} \Rightarrow \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4}$$

Now, we have:

$$\begin{aligned}
 \text{L.H.S.} &= 2 \sin^{-1} \frac{3}{5} = 2 \tan^{-1} \frac{3}{4} \\
 &= \tan^{-1} \left(\frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} \right) \quad \left[2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \right] \\
 &= \tan^{-1} \left(\frac{\frac{3}{2}}{\frac{16-9}{16}} \right) = \tan^{-1} \left(\frac{3}{2} \times \frac{16}{7} \right) \\
 &= \tan^{-1} \frac{24}{7} = \text{R.H.S.}
 \end{aligned}$$

Question 4:

Prove $\sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{77}{36}$

Answer

Let $\sin^{-1} \frac{8}{17} = x$. Then, $\sin x = \frac{8}{17} \Rightarrow \cos x = \sqrt{1 - \left(\frac{8}{17}\right)^2} = \sqrt{\frac{225}{289}} = \frac{15}{17}$.

$$\therefore \tan x = \frac{8}{15} \Rightarrow x = \tan^{-1} \frac{8}{15}$$

$$\therefore \sin^{-1} \frac{8}{17} = \tan^{-1} \frac{8}{15} \quad \dots(1)$$

Now, let $\sin^{-1} \frac{3}{5} = y$. Then, $\sin y = \frac{3}{5} \Rightarrow \cos y = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

$$\therefore \tan y = \frac{3}{4} \Rightarrow y = \tan^{-1} \frac{3}{4}$$

$$\therefore \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \quad \dots(2)$$

Now, we have:

$$\begin{aligned}\text{L.H.S.} &= \sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} \\ &= \tan^{-1} \frac{8}{15} + \tan^{-1} \frac{3}{4} && \text{[Using (1) and (2)]} \\ &= \tan^{-1} \frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15} \times \frac{3}{4}} \\ &= \tan^{-1} \left(\frac{32 + 45}{60 - 24} \right) && \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\ &= \tan^{-1} \frac{77}{36} = \text{R.H.S.}\end{aligned}$$

Question 5:

Prove $\cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65}$

Answer

$$\text{Let } \cos^{-1} \frac{4}{5} = x. \text{ Then, } \cos x = \frac{4}{5} \Rightarrow \sin x = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}.$$

$$\therefore \tan x = \frac{3}{4} \Rightarrow x = \tan^{-1} \frac{3}{4}$$

$$\therefore \cos^{-1} \frac{4}{5} = \tan^{-1} \frac{3}{4} \quad \dots(1)$$

$$\text{Now, let } \cos^{-1} \frac{12}{13} = y. \text{ Then, } \cos y = \frac{12}{13} \Rightarrow \sin y = \frac{5}{13}.$$

$$\therefore \tan y = \frac{5}{12} \Rightarrow y = \tan^{-1} \frac{5}{12}$$

$$\therefore \cos^{-1} \frac{12}{13} = \tan^{-1} \frac{5}{12} \quad \dots(2)$$

$$\text{Let } \cos^{-1} \frac{33}{65} = z. \text{ Then, } \cos z = \frac{33}{65} \Rightarrow \sin z = \frac{56}{65}.$$

$$\therefore \tan z = \frac{56}{33} \Rightarrow z = \tan^{-1} \frac{56}{33}$$

$$\therefore \cos^{-1} \frac{33}{65} = \tan^{-1} \frac{56}{33} \quad \dots(3)$$

Now, we will prove that:

$$\text{L.H.S.} = \cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13}$$

$$= \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{5}{12} \quad [\text{Using (1) and (2)}]$$

$$= \tan^{-1} \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right]$$

$$= \tan^{-1} \frac{36+20}{48-15}$$

$$= \tan^{-1} \frac{56}{33}$$

$$= \tan^{-1} \frac{56}{33} \quad [\text{by (3)}]$$

$$= \text{R.H.S.}$$

Question 6:

Prove $\cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{56}{65}$

Answer

Let $\sin^{-1} \frac{3}{5} = x$. Then, $\sin x = \frac{3}{5} \Rightarrow \cos x = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

$$\therefore \tan x = \frac{3}{4} \Rightarrow x = \tan^{-1} \frac{3}{4}$$

$$\therefore \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \quad \dots(1)$$

Now, let $\cos^{-1} \frac{12}{13} = y$. Then, $\cos y = \frac{12}{13} \Rightarrow \sin y = \frac{5}{13}$.

$$\therefore \tan y = \frac{5}{12} \Rightarrow y = \tan^{-1} \frac{5}{12}$$

$$\therefore \cos^{-1} \frac{12}{13} = \tan^{-1} \frac{5}{12} \quad \dots(2)$$

Let $\sin^{-1} \frac{56}{65} = z$. Then, $\sin z = \frac{56}{65} \Rightarrow \cos z = \frac{33}{65}$.

$$\therefore \tan z = \frac{56}{33} \Rightarrow z = \tan^{-1} \frac{56}{33}$$

$$\therefore \sin^{-1} \frac{56}{65} = \tan^{-1} \frac{56}{33} \quad \dots(3)$$

Now, we have:

$$\begin{aligned}
 \text{L.H.S.} &= \cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} \\
 &= \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{3}{4} && \text{[Using (1) and (2)]} \\
 &= \tan^{-1} \frac{\frac{5}{12} + \frac{3}{4}}{1 - \frac{5}{12} \cdot \frac{3}{4}} && \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\
 &= \tan^{-1} \frac{20+36}{48-15} \\
 &= \tan^{-1} \frac{56}{33} \\
 &= \sin^{-1} \frac{56}{65} = \text{R.H.S.} && \text{[Using (3)]}
 \end{aligned}$$

Question 7:

Prove $\tan^{-1} \frac{63}{16} = \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5}$

Answer

Let $\sin^{-1} \frac{5}{13} = x$. Then, $\sin x = \frac{5}{13} \Rightarrow \cos x = \frac{12}{13}$.

$\therefore \tan x = \frac{5}{12} \Rightarrow x = \tan^{-1} \frac{5}{12}$

$\therefore \sin^{-1} \frac{5}{13} = \tan^{-1} \frac{5}{12} \quad \dots(1)$

Let $\cos^{-1} \frac{3}{5} = y$. Then, $\cos y = \frac{3}{5} \Rightarrow \sin y = \frac{4}{5}$.

$\therefore \tan y = \frac{4}{3} \Rightarrow y = \tan^{-1} \frac{4}{3}$

$\therefore \cos^{-1} \frac{3}{5} = \tan^{-1} \frac{4}{3} \quad \dots(2)$

Using (1) and (2), we have

$$\begin{aligned}\text{R.H.S.} &= \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5} \\ &= \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{4}{3} \\ &= \tan^{-1} \left(\frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \times \frac{4}{3}} \right) \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\ &= \tan^{-1} \left(\frac{15+48}{36-20} \right) \\ &= \tan^{-1} \frac{63}{16} \\ &= \text{L.H.S.}\end{aligned}$$

Question 8:

Prove $\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$

Answer

$$\begin{aligned}
 \text{L.H.S.} &= \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} \\
 &= \tan^{-1} \left(\frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \times \frac{1}{7}} \right) + \tan^{-1} \left(\frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \times \frac{1}{8}} \right) \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\
 &= \tan^{-1} \left(\frac{7+5}{35-1} \right) + \tan^{-1} \left(\frac{8+3}{24-1} \right) \\
 &= \tan^{-1} \frac{12}{34} + \tan^{-1} \frac{11}{23} \\
 &= \tan^{-1} \frac{6}{17} + \tan^{-1} \frac{11}{23} \\
 &= \tan^{-1} \left(\frac{\frac{6}{17} + \frac{11}{23}}{1 - \frac{6}{17} \times \frac{11}{23}} \right) \\
 &= \tan^{-1} \left(\frac{138+187}{391-66} \right) \\
 &= \tan^{-1} \left(\frac{325}{325} \right) = \tan^{-1} 1 \\
 &= \frac{\pi}{4} = \text{R.H.S.}
 \end{aligned}$$

Question 9:

$$\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right), \quad x \in [0, 1]$$

Prove

Answer

$$\text{Let } x = \tan^2 \theta. \text{ Then, } \sqrt{x} = \tan \theta \Rightarrow \theta = \tan^{-1} \sqrt{x}.$$

$$\therefore \frac{1-x}{1+x} = \frac{1-\tan^2 \theta}{1+\tan^2 \theta} = \cos 2\theta$$

Now, we have:

$$\text{R.H.S.} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right) = \frac{1}{2} \cos^{-1} (\cos 2\theta) = \frac{1}{2} \times 2\theta = \theta = \tan^{-1} \sqrt{x} = \text{L.H.S.}$$

Question 10:

Prove $\cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4} \right)$

Answer

Consider $\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}$

$$= \frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2 - (\sqrt{1-\sin x})^2} \quad (\text{by rationalizing})$$

$$= \frac{(1+\sin x) + (1-\sin x) + 2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x - 1+\sin x}$$

$$= \frac{2(1+\sqrt{1-\sin^2 x})}{2\sin x} = \frac{1+\cos x}{\sin x} = \frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \cot \frac{x}{2}$$

\therefore L.H.S. = $\cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \cot^{-1} \left(\cot \frac{x}{2} \right) = \frac{x}{2} = \text{R.H.S.}$

Question 11:

Prove $\tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq 1$ [Hint: put $x = \cos 2\theta$]

Answer

Put $x = \cos 2\theta$ so that $\theta = \frac{1}{2} \cos^{-1} x$. Then, we have:

$$\begin{aligned} \text{L.H.S.} &= \tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) \\ &= \tan^{-1} \left(\frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}} \right) \\ &= \tan^{-1} \left(\frac{\sqrt{2\cos^2 \theta} - \sqrt{2\sin^2 \theta}}{\sqrt{2\cos^2 \theta} + \sqrt{2\sin^2 \theta}} \right) \\ &= \tan^{-1} \left(\frac{\sqrt{2}\cos \theta - \sqrt{2}\sin \theta}{\sqrt{2}\cos \theta + \sqrt{2}\sin \theta} \right) \\ &= \tan^{-1} \left(\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right) = \tan^{-1} \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right) \\ &= \tan^{-1} 1 - \tan^{-1} (\tan \theta) \\ &= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x = \text{R.H.S.} \end{aligned}$$

$$\left[\tan^{-1} \left(\frac{x-y}{1+xy} \right) = \tan^{-1} x - \tan^{-1} y \right]$$

Question 12:

Prove $\frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \frac{1}{3} = \frac{9}{4} \sin^{-1} \frac{2\sqrt{2}}{3}$

Answer

$$\begin{aligned} \text{L.H.S.} &= \frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \frac{1}{3} \\ &= \frac{9}{4} \left(\frac{\pi}{2} - \sin^{-1} \frac{1}{3} \right) \\ &= \frac{9}{4} \left(\cos^{-1} \frac{1}{3} \right) \quad \dots(1) \quad \left[\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right] \end{aligned}$$

Now, let $\cos^{-1} \frac{1}{3} = x$. Then, $\cos x = \frac{1}{3} \Rightarrow \sin x = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$.

$$\therefore x = \sin^{-1} \frac{2\sqrt{2}}{3} \Rightarrow \cos^{-1} \frac{1}{3} = \sin^{-1} \frac{2\sqrt{2}}{3}$$

$$\therefore \text{L.H.S.} = \frac{9}{4} \sin^{-1} \frac{2\sqrt{2}}{3} = \text{R.H.S.}$$

Question 13:

Solve $2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$

Answer

$$2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1} \left(\frac{2 \cos x}{1 - \cos^2 x} \right) = \tan^{-1}(2 \operatorname{cosec} x) \quad \left[2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \right]$$

$$\Rightarrow \frac{2 \cos x}{1 - \cos^2 x} = 2 \operatorname{cosec} x$$

$$\Rightarrow \frac{2 \cos x}{\sin^2 x} = \frac{2}{\sin x}$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\therefore x = \frac{\pi}{4}$$

Question 14:

Solve $\tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x, (x > 0)$

Answer

$$\tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x$$

$$\Rightarrow \tan^{-1} 1 - \tan^{-1} x = \frac{1}{2} \tan^{-1} x \quad \left[\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy} \right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{3}{2} \tan^{-1} x$$

$$\Rightarrow \tan^{-1} x = \frac{\pi}{6}$$

$$\Rightarrow x = \tan \frac{\pi}{6}$$

$$\therefore x = \frac{1}{\sqrt{3}}$$

Question 15:

Solve $\sin(\tan^{-1} x)$, $|x| < 1$ is equal to

- (A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

Answer

Let $\tan^{-1} x = y$. Then, $\tan y = x \Rightarrow \sin y = \frac{x}{\sqrt{1+x^2}}$.

$$\therefore y = \sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right) \Rightarrow \tan^{-1} x = \sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right)$$

$$\therefore \sin(\tan^{-1} x) = \sin \left(\sin^{-1} \frac{x}{\sqrt{1+x^2}} \right) = \frac{x}{\sqrt{1+x^2}}$$

The correct answer is D.

Question 16:

Solve $\sin^{-1}(1-x) - 2 \sin^{-1} x = \frac{\pi}{2}$, then x is equal to

$$\text{(A) } 0, \frac{1}{2} \quad \text{(B) } 1, \frac{1}{2} \quad \text{(C) } 0 \quad \text{(D) } \frac{1}{2}$$

Answer

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow -2\sin^{-1}x = \frac{\pi}{2} - \sin^{-1}(1-x)$$

$$\Rightarrow -2\sin^{-1}x = \cos^{-1}(1-x) \quad \dots(1)$$

$$\text{Let } \sin^{-1}x = \theta \Rightarrow \sin\theta = x \Rightarrow \cos\theta = \sqrt{1-x^2}.$$

$$\therefore \theta = \cos^{-1}(\sqrt{1-x^2})$$

$$\therefore \sin^{-1}x = \cos^{-1}(\sqrt{1-x^2})$$

Therefore, from equation (1), we have

$$-2\cos^{-1}(\sqrt{1-x^2}) = \cos^{-1}(1-x)$$

Put $x = \sin y$. Then, we have:

$$-2\cos^{-1}(\sqrt{1-\sin^2 y}) = \cos^{-1}(1-\sin y)$$

$$\Rightarrow -2\cos^{-1}(\cos y) = \cos^{-1}(1-\sin y)$$

$$\Rightarrow -2y = \cos^{-1}(1-\sin y)$$

$$\Rightarrow 1-\sin y = \cos(-2y) = \cos 2y$$

$$\Rightarrow 1-\sin y = 1-2\sin^2 y$$

$$\Rightarrow 2\sin^2 y - \sin y = 0$$

$$\Rightarrow \sin y(2\sin y - 1) = 0$$

$$\Rightarrow \sin y = 0 \text{ or } \frac{1}{2}$$

$$\therefore x = 0 \text{ or } x = \frac{1}{2}$$

But, when $x = \frac{1}{2}$, it can be observed that:

$$\begin{aligned}\text{L.H.S.} &= \sin^{-1}\left(1 - \frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2} \\ &= \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2} \\ &= -\sin^{-1}\frac{1}{2} \\ &= -\frac{\pi}{6} \neq \frac{\pi}{2} \neq \text{R.H.S.}\end{aligned}$$

$\therefore x = \frac{1}{2}$ is not the solution of the given equation.

Thus, $x = 0$.

Hence, the correct answer is **C**.

Question 17:

Solve $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{-3\pi}{4}$

Answer

$$\begin{aligned} & \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y} \\ &= \tan^{-1}\left[\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right] && \left[\tan^{-1} y - \tan^{-1} x = \tan^{-1} \frac{y-x}{1+xy}\right] \\ &= \tan^{-1}\left[\frac{\frac{x(x+y) - y(x-y)}{y(x+y)}}{\frac{y(x+y) + x(x-y)}{y(x+y)}}\right] \\ &= \tan^{-1}\left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right) \\ &= \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + y^2}\right) = \tan^{-1} 1 = \frac{\pi}{4} \end{aligned}$$

Hence, the correct answer is **C**.