

## Exercise 3.3

## Question 1:

Find the transpose of each of the following matrices:

$$(i) \begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix} \quad (ii) \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \quad (iii) \begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$$

Answer

$$(i) \text{ Let } A = \begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$$

$$(ii) \text{ Let } A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$(iii) \text{ Let } A = \begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$$

## Question 2:

$$\text{If } A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}, \text{ then verify that}$$

$$(i) (A+B)' = A' + B'$$

$$(ii) (A-B)' = A' - B'$$

Answer

We have:

$$A' = \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix}, B' = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix}$$

(i)

$$A+B = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 3 & -2 \\ 6 & 9 & 9 \\ -1 & 4 & 2 \end{bmatrix}$$

$$\therefore (A+B)' = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix}$$

$$A'+B' = \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix}$$

Hence, we have verified that  $(A+B)' = A'+B'$

(ii)

$$A-B = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 8 \\ 4 & 5 & 9 \\ -3 & -2 & 0 \end{bmatrix}$$

$$\therefore (A-B)' = \begin{bmatrix} 3 & 4 & -3 \\ 1 & 5 & -2 \\ 8 & 9 & 0 \end{bmatrix}$$

$$A'-B' = \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -3 \\ 1 & 5 & -2 \\ 8 & 9 & 0 \end{bmatrix}$$

Hence, we have verified that  $(A-B)' = A'-B'$ .

**Question 3:**

If  $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ , then verify that

(i)  $(A+B)' = A' + B'$

(ii)  $(A-B)' = A' - B'$

Answer

**(i)** It is known that  $A = (A')'$

Therefore, we have:

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$B' = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 4 & 4 \end{bmatrix}$$

$$\therefore (A+B)' = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

$$A' + B' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix}$$

Thus, we have verified that  $(A+B)' = A' + B'$ .

(ii)

$$A - B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -3 & -1 \\ 3 & 0 & -2 \end{bmatrix}$$

$$\therefore (A - B)' = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$$

$$A' - B' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$$

Thus, we have verified that  $(A - B)' = A' - B'$ .

**Question 4:**

If  $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$ , then find  $(A + 2B)'$

Answer

We know that  $A = (A)'$

$$\therefore A = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\therefore A + 2B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} + 2 \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}$$

$$\therefore (A + 2B)' = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$$

**Question 5:**

For the matrices  $A$  and  $B$ , verify that  $(AB)' = B'A'$  where

$$(i) \quad A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, B = [-1 \quad 2 \quad 1]$$

$$(ii) \quad A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, B = [1 \quad 5 \quad 7]$$

Answer

**(i)**

$$AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} [-1 \quad 2 \quad 1] = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$$

$$\therefore (AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

$$\text{Now, } A' = [1 \quad -4 \quad 3], B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\therefore B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} [1 \quad -4 \quad 3] = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

Hence, we have verified that  $(AB)' = B'A'$ .

**(ii)**

$$AB = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}$$

$$\therefore (AB)' = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$$

$$\text{Now, } A' = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}, B' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$$

$$\therefore B'A' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix}$$

Hence, we have verified that  $(AB)' = B'A'$ .

**Question 6:**

If (i)  $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$ , then verify that  $A'A = I$

(ii)  $A = \begin{bmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{bmatrix}$ , then verify that  $A'A = I$

Answer

**(i)**

$$A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\therefore A' = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$\begin{aligned} A'A &= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} (\cos \alpha)(\cos \alpha) + (-\sin \alpha)(-\sin \alpha) & (\cos \alpha)(\sin \alpha) + (-\sin \alpha)(\cos \alpha) \\ (\sin \alpha)(\cos \alpha) + (\cos \alpha)(-\sin \alpha) & (\sin \alpha)(\sin \alpha) + (\cos \alpha)(\cos \alpha) \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \alpha + \sin^2 \alpha & \sin \alpha \cos \alpha - \sin \alpha \cos \alpha \\ \sin \alpha \cos \alpha - \sin \alpha \cos \alpha & \sin^2 \alpha + \cos^2 \alpha \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

Hence, we have verified that  $A'A = I$ .

**(ii)**

$$A = \begin{bmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{bmatrix}$$

$$\therefore A' = \begin{bmatrix} \sin \alpha & -\cos \alpha \\ \cos \alpha & \sin \alpha \end{bmatrix}$$

$$A'A = \begin{bmatrix} \sin \alpha & -\cos \alpha \\ \cos \alpha & \sin \alpha \end{bmatrix} \begin{bmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{bmatrix}$$

$$\begin{aligned}
 & \begin{bmatrix} \sin \alpha & -\cos \alpha \\ \cos \alpha & \sin \alpha \end{bmatrix} \begin{bmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{bmatrix} \\
 &= \begin{bmatrix} (\sin \alpha)(\sin \alpha) + (-\cos \alpha)(-\cos \alpha) & (\sin \alpha)(\cos \alpha) + (-\cos \alpha)(\sin \alpha) \\ (\cos \alpha)(\sin \alpha) + (\sin \alpha)(-\cos \alpha) & (\cos \alpha)(\cos \alpha) + (\sin \alpha)(\sin \alpha) \end{bmatrix} \\
 &= \begin{bmatrix} \sin^2 \alpha + \cos^2 \alpha & \sin \alpha \cos \alpha - \sin \alpha \cos \alpha \\ \sin \alpha \cos \alpha - \sin \alpha \cos \alpha & \cos^2 \alpha + \sin^2 \alpha \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
 \end{aligned}$$

Hence, we have verified that  $A'A = I$ .

**Question 7:**

(i) Show that the matrix  $A = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$  is a symmetric matrix

(ii) Show that the matrix  $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$  is a skew symmetric matrix

Answer

**(i)** We have:

$$A' = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix} = A$$

$$\therefore A' = A$$

Hence,  $A$  is a symmetric matrix.

**(ii)** We have:

$$A' = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = -A$$

$$\therefore A' = -A$$

Hence,  $A$  is a skew-symmetric matrix.

**Question 8:**

For the matrix  $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$ , verify that

(i)  $(A + A')$  is a symmetric matrix

(ii)  $(A - A')$  is a skew symmetric matrix

Answer

$$A' = \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix}$$

$$(i) \quad A + A' = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}$$

$$\therefore (A + A')' = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix} = A + A'$$

Hence,  $(A + A')$  is a symmetric matrix.

$$(ii) \quad A - A' = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(A - A')' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -(A - A')$$

Hence,  $(A - A')$  is a skew-symmetric matrix.

**Question 9:**

Find  $\frac{1}{2}(A+A')$  and  $\frac{1}{2}(A-A')$ , when  $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

Answer

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

The given matrix is

$$\therefore A' = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

$$A + A' = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \frac{1}{2}(A + A') = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Now, } A - A' = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2a & 2b \\ -2a & 0 & 2c \\ -2b & -2c & 0 \end{bmatrix}$$

$$\therefore \frac{1}{2}(A - A') = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

**Question 10:**

Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

(i)  $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$

$$(ii) \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

$$(iv) \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$$

Answer

**(i)**

$$\text{Let } A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$$

$$\text{Now, } A + A' = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}(A + A') = \frac{1}{2} \begin{bmatrix} 6 & 6 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}$$

$$\text{Now, } P' = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} = P$$

Thus,  $P = \frac{1}{2}(A + A')$  is a symmetric matrix.

$$\text{Now, } A - A' = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

$$\text{Let } Q = \frac{1}{2}(A - A') = \frac{1}{2} \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$\text{Now, } Q' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = -Q$$

Thus,  $Q = \frac{1}{2}(A - A')$  is a skew-symmetric matrix.

Representing  $A$  as the sum of  $P$  and  $Q$ :

$$P + Q = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = A$$

(ii)

$$\text{Let } A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\text{Now, } A + A' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}(A + A') = \frac{1}{2} \begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\text{Now, } P' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = P$$

Thus,  $P = \frac{1}{2}(A + A')$  is a symmetric matrix.

$$\text{Now, } A - A' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Let } Q = \frac{1}{2}(A - A') = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Now, } Q' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -Q$$

Thus,  $Q = \frac{1}{2}(A - A')$  is a skew-symmetric matrix.

Representing  $A$  as the sum of  $P$  and  $Q$ :

$$P + Q = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = A$$

(iii)

$$\text{Let } A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\text{Now, } A + A' = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}(A + A') = \frac{1}{2} \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}$$

$$\text{Now, } P' = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix} = P$$

Thus,  $P = \frac{1}{2}(A + A')$  is a symmetric matrix.

$$\text{Now, } A - A' = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

$$\text{Let } Q = \frac{1}{2}(A - A') = \frac{1}{2} \begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ -\frac{3}{2} & -3 & 0 \end{bmatrix}$$

$$\text{Now, } Q' = \begin{bmatrix} 0 & -\frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix} = -Q$$

Thus,  $Q = \frac{1}{2}(A - A')$  is a skew-symmetric matrix.

Representing  $A$  as the sum of  $P$  and  $Q$ :

$$P+Q = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ -\frac{3}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} = A$$

(iv)

$$\text{Let } A = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}$$

$$\text{Now } A + A' = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}(A + A') = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\text{Now, } P' = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = P$$

Thus,  $P = \frac{1}{2}(A + A')$  is a symmetric matrix.

$$\text{Now, } A - A' = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}$$

$$\text{Let } Q = \frac{1}{2}(A - A') = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

$$\text{Now, } Q' = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = -Q$$

Thus,  $Q = \frac{1}{2}(A - A')$  is a skew-symmetric matrix.

Representing  $A$  as the sum of  $P$  and  $Q$ :

$$P+Q = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} = A$$

**Question 11:**

If  $A, B$  are symmetric matrices of same order, then  $AB - BA$  is a

- A.** Skew symmetric matrix **B.** Symmetric matrix  
**C.** Zero matrix **D.** Identity matrix

Answer

The correct answer is A.

$A$  and  $B$  are symmetric matrices, therefore, we have:

$$A' = A \text{ and } B' = B \quad \dots(1)$$

$$\begin{aligned} \text{Consider } (AB - BA)' &= (AB)' - (BA)' && \left[ (A - B)' = A' - B' \right] \\ &= B'A' - A'B' && \left[ (AB)' = B'A' \right] \\ &= BA - AB && \left[ \text{by (1)} \right] \\ &= -(AB - BA) \end{aligned}$$

$$\therefore (AB - BA)' = -(AB - BA)$$

Thus,  $(AB - BA)$  is a skew-symmetric matrix.

**Question 12:**

If  $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$ , then  $A + A' = I$ , if the value of  $\alpha$  is

- A.**  $\frac{\pi}{6}$  **B.**  $\frac{\pi}{3}$   
**C.**  $\frac{\pi}{2}$  **D.**  $\frac{3\pi}{2}$

Answer

The correct answer is B.

$$A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$\Rightarrow A' = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$$

Now,  $A + A' = I$

$$\therefore \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} + \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\cos \alpha & 0 \\ 0 & 2\cos \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Comparing the corresponding elements of the two matrices, we have:

$$2\cos \alpha = 1$$

$$\Rightarrow \cos \alpha = \frac{1}{2} = \cos \frac{\pi}{3}$$

$$\therefore \alpha = \frac{\pi}{3}$$

#### Exercise 3.4

##### Question 1:

Find the inverse of each of the matrices, if it exists.

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

We know that  $A = IA$