

Exercise 5.7**Question 1:**

Find the second order derivatives of the function.

$$x^2 + 3x + 2$$

Answer

Let $y = x^2 + 3x + 2$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 2x + 3 + 0 = 2x + 3$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(2x + 3) = \frac{d}{dx}(2x) + \frac{d}{dx}(3) = 2 + 0 = 2$$

Question 2:

Find the second order derivatives of the function.

$$x^{20}$$

Answer

Let $y = x^{20}$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x^{20}) = 20x^{19}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}(20x^{19}) = 20 \frac{d}{dx}(x^{19}) = 20 \cdot 19 \cdot x^{18} = 380x^{18}$$

Question 3:

Find the second order derivatives of the function.

$$x \cdot \cos x$$

Answer

Let $y = x \cdot \cos x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot \cos x) = \cos x \cdot \frac{d}{dx}(x) + x \frac{d}{dx}(\cos x) = \cos x \cdot 1 + x(-\sin x) = \cos x - x \sin x$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[\cos x - x \sin x] = \frac{d}{dx}(\cos x) - \frac{d}{dx}(x \sin x)$$

$$= -\sin x - \left[\sin x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\sin x) \right]$$

$$= -\sin x - (\sin x + x \cos x)$$

$$= -(x \cos x + 2 \sin x)$$

Question 4:

Find the second order derivatives of the function.

$\log x$

Answer

Let $y = \log x$

Then,

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

Question 5:

Find the second order derivatives of the function.

$x^3 \log x$

Answer

Let $y = x^3 \log x$

Then,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [x^3 \log x] = \log x \cdot \frac{d}{dx} (x^3) + x^3 \cdot \frac{d}{dx} (\log x) \\ &= \log x \cdot 3x^2 + x^3 \cdot \frac{1}{x} = \log x \cdot 3x^2 + x^2 \\ &= x^2 (1 + 3 \log x) \\ \therefore \frac{d^2 y}{dx^2} &= \frac{d}{dx} [x^2 (1 + 3 \log x)] \\ &= (1 + 3 \log x) \cdot \frac{d}{dx} (x^2) + x^2 \cdot \frac{d}{dx} (1 + 3 \log x) \\ &= (1 + 3 \log x) \cdot 2x + x^2 \cdot \frac{3}{x} \\ &= 2x + 6x \log x + 3x \\ &= 5x + 6x \log x \\ &= x(5 + 6 \log x) \end{aligned}$$

Question 6:

Find the second order derivatives of the function.

$$e^x \sin 5x$$

Answer

$$\text{Let } y = e^x \sin 5x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} (e^x \sin 5x) = \sin 5x \cdot \frac{d}{dx} (e^x) + e^x \frac{d}{dx} (\sin 5x) \\ &= \sin 5x \cdot e^x + e^x \cdot \cos 5x \cdot \frac{d}{dx} (5x) = e^x \sin 5x + e^x \cos 5x \cdot 5 \\ &= e^x (\sin 5x + 5 \cos 5x) \end{aligned}$$

$$\begin{aligned} \therefore \frac{d^2 y}{dx^2} &= \frac{d}{dx} [e^x (\sin 5x + 5 \cos 5x)] \\ &= (\sin 5x + 5 \cos 5x) \cdot \frac{d}{dx} (e^x) + e^x \cdot \frac{d}{dx} (\sin 5x + 5 \cos 5x) \\ &= (\sin 5x + 5 \cos 5x) e^x + e^x \left[\cos 5x \cdot \frac{d}{dx} (5x) + 5(-\sin 5x) \cdot \frac{d}{dx} (5x) \right] \\ &= e^x (\sin 5x + 5 \cos 5x) + e^x (5 \cos 5x - 25 \sin 5x) \\ &= e^x (10 \cos 5x - 24 \sin 5x) = 2e^x (5 \cos 5x - 12 \sin 5x) \end{aligned}$$

Then,

Question 7:

Find the second order derivatives of the function.

$$e^{6x} \cos 3x$$

Answer

$$\text{Let } y = e^{6x} \cos 3x$$

Then,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(e^{6x} \cdot \cos 3x) = \cos 3x \cdot \frac{d}{dx}(e^{6x}) + e^{6x} \cdot \frac{d}{dx}(\cos 3x) \\ &= \cos 3x \cdot e^{6x} \cdot \frac{d}{dx}(6x) + e^{6x} \cdot (-\sin 3x) \cdot \frac{d}{dx}(3x) \\ &= 6e^{6x} \cos 3x - 3e^{6x} \sin 3x \quad \dots(1) \end{aligned}$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} &= \frac{d}{dx}(6e^{6x} \cos 3x - 3e^{6x} \sin 3x) = 6 \cdot \frac{d}{dx}(e^{6x} \cos 3x) - 3 \cdot \frac{d}{dx}(e^{6x} \sin 3x) \\ &= 6 \cdot [6e^{6x} \cos 3x - 3e^{6x} \sin 3x] - 3 \cdot \left[\sin 3x \cdot \frac{d}{dx}(e^{6x}) + e^{6x} \cdot \frac{d}{dx}(\sin 3x) \right] \quad [\text{Using (1)}] \\ &= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 3 \left[\sin 3x \cdot e^{6x} \cdot 6 + e^{6x} \cdot \cos 3x \cdot 3 \right] \\ &= 36e^{6x} \cos 3x - 18e^{6x} \sin 3x - 18e^{6x} \sin 3x - 9e^{6x} \cos 3x \\ &= 27e^{6x} \cos 3x - 36e^{6x} \sin 3x \\ &= 9e^{6x} (3 \cos 3x - 4 \sin 3x) \end{aligned}$$

Question 8:

Find the second order derivatives of the function.

$$\tan^{-1} x$$

Answer

$$\text{Let } y = \tan^{-1} x$$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \\ \therefore \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{d}{dx}(1+x^2)^{-1} = (-1) \cdot (1+x^2)^{-2} \cdot \frac{d}{dx}(1+x^2) \\ &= \frac{-1}{(1+x^2)^2} \times 2x = \frac{-2x}{(1+x^2)^2}\end{aligned}$$

Question 9:

Find the second order derivatives of the function.

$$\log(\log x)$$

Answer

$$\text{Let } y = \log(\log x)$$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[\log(\log x)] = \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) = \frac{1}{x \log x} = (x \log x)^{-1} \\ \therefore \frac{d^2y}{dx^2} &= \frac{d}{dx}[(x \log x)^{-1}] = (-1) \cdot (x \log x)^{-2} \cdot \frac{d}{dx}(x \log x) \\ &= \frac{-1}{(x \log x)^2} \cdot \left[\log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x) \right] \\ &= \frac{-1}{(x \log x)^2} \cdot \left[\log x \cdot 1 + x \cdot \frac{1}{x} \right] = \frac{-(1 + \log x)}{(x \log x)^2}\end{aligned}$$

Question 10:

Find the second order derivatives of the function.

$$\sin(\log x)$$

Answer

$$\text{Let } y = \sin(\log x)$$

Then,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [\sin(\log x)] = \cos(\log x) \cdot \frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x} \\ \therefore \frac{d^2y}{dx^2} &= \frac{d}{dx} \left[\frac{\cos(\log x)}{x} \right] \\ &= \frac{x \cdot \frac{d}{dx} [\cos(\log x)] - \cos(\log x) \cdot \frac{d}{dx}(x)}{x^2} \\ &= \frac{x \cdot \left[-\sin(\log x) \cdot \frac{d}{dx}(\log x) \right] - \cos(\log x) \cdot 1}{x^2} \\ &= \frac{-x \sin(\log x) \cdot \frac{1}{x} - \cos(\log x)}{x^2} \\ &= \frac{-[\sin(\log x) + \cos(\log x)]}{x^2} \end{aligned}$$

Question 11:

If $y = 5 \cos x - 3 \sin x$, prove that $\frac{d^2y}{dx^2} + y = 0$

Answer

It is given that, $y = 5 \cos x - 3 \sin x$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(5 \cos x) - \frac{d}{dx}(3 \sin x) = 5 \frac{d}{dx}(\cos x) - 3 \frac{d}{dx}(\sin x) \\ &= 5(-\sin x) - 3 \cos x = -(5 \sin x + 3 \cos x)\end{aligned}$$

$$\begin{aligned}\therefore \frac{d^2y}{dx^2} &= \frac{d}{dx}[-(5 \sin x + 3 \cos x)] \\ &= -\left[5 \cdot \frac{d}{dx}(\sin x) + 3 \cdot \frac{d}{dx}(\cos x)\right] \\ &= -[5 \cos x + 3(-\sin x)] \\ &= -[5 \cos x - 3 \sin x] \\ &= -y\end{aligned}$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

Hence, proved.

Question 12:

If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.

Answer

It is given that, $y = \cos^{-1} x$

Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} = -(1-x^2)^{-\frac{1}{2}} \\ \frac{d^2y}{dx^2} &= \frac{d}{dx} \left[-(1-x^2)^{-\frac{1}{2}} \right] \\ &= - \left(-\frac{1}{2} \right) \cdot (1-x^2)^{-\frac{3}{2}} \cdot \frac{d}{dx}(1-x^2) \\ &= \frac{1}{2\sqrt{(1-x^2)^3}} \times (-2x) \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{-x}{\sqrt{(1-x^2)^3}} \quad \dots(i)\end{aligned}$$

$$y = \cos^{-1} x \Rightarrow x = \cos y$$

Putting $x = \cos y$ in equation (i), we obtain

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{-\cos y}{\sqrt{(1-\cos^2 y)^3}} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{-\cos y}{\sqrt{(\sin^2 y)^3}} \\ &= \frac{-\cos y}{\sin^3 y} \\ &= \frac{-\cos y}{\sin y} \times \frac{1}{\sin^2 y} \\ \Rightarrow \frac{d^2y}{dx^2} &= -\cot y \cdot \operatorname{cosec}^2 y\end{aligned}$$

Question 13:

If $y = 3 \cos(\log x) + 4 \sin(\log x)$, show that $x^2 y_2 + x y_1 + y = 0$

Answer

It is given that, $y = 3 \cos(\log x) + 4 \sin(\log x)$

Then,

$$\begin{aligned}
 y_1 &= 3 \cdot \frac{d}{dx} [\cos(\log x)] + 4 \cdot \frac{d}{dx} [\sin(\log x)] \\
 &= 3 \cdot \left[-\sin(\log x) \cdot \frac{d}{dx} (\log x) \right] + 4 \cdot \left[\cos(\log x) \cdot \frac{d}{dx} (\log x) \right] \\
 \therefore y_1 &= \frac{-3 \sin(\log x)}{x} + \frac{4 \cos(\log x)}{x} = \frac{4 \cos(\log x) - 3 \sin(\log x)}{x} \\
 \therefore y_2 &= \frac{d}{dx} \left(\frac{4 \cos(\log x) - 3 \sin(\log x)}{x} \right) \\
 &= \frac{x \{4 \cos(\log x) - 3 \sin(\log x)\}' - \{4 \cos(\log x) - 3 \sin(\log x)\} (x)'}{x^2} \\
 &= \frac{x \left[4 \{\cos(\log x)\}' - 3 \{\sin(\log x)\}' \right] - \{4 \cos(\log x) - 3 \sin(\log x)\} \cdot 1}{x^2} \\
 &= \frac{x \left[-4 \sin(\log x) \cdot (\log x)' - 3 \cos(\log x) \cdot (\log x)' \right] - 4 \cos(\log x) + 3 \sin(\log x)}{x^2} \\
 &= \frac{x \left[-4 \sin(\log x) \cdot \frac{1}{x} - 3 \cos(\log x) \cdot \frac{1}{x} \right] - 4 \cos(\log x) + 3 \sin(\log x)}{x^2} \\
 &= \frac{-4 \sin(\log x) - 3 \cos(\log x) - 4 \cos(\log x) + 3 \sin(\log x)}{x^2} \\
 &= \frac{-\sin(\log x) - 7 \cos(\log x)}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \therefore x^2 y_2 + x y_1 + y &= x^2 \left(\frac{-\sin(\log x) - 7 \cos(\log x)}{x^2} \right) + x \left(\frac{4 \cos(\log x) - 3 \sin(\log x)}{x} \right) + 3 \cos(\log x) + 4 \sin(\log x) \\
 &= -\sin(\log x) - 7 \cos(\log x) + 4 \cos(\log x) - 3 \sin(\log x) + 3 \cos(\log x) + 4 \sin(\log x) \\
 &= 0
 \end{aligned}$$

Hence, proved.

Question 14:

If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2 y}{dx^2} - (m+n) \frac{dy}{dx} + mny = 0$

Answer

It is given that, $y = Ae^{mx} + Be^{nx}$

Then,

$$\frac{dy}{dx} = A \cdot \frac{d}{dx}(e^{mx}) + B \cdot \frac{d}{dx}(e^{nx}) = A \cdot e^{mx} \cdot \frac{d}{dx}(mx) + B \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Ame^{mx} + Bne^{nx}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx}(Ame^{mx} + Bne^{nx}) = Am \cdot \frac{d}{dx}(e^{mx}) + Bn \cdot \frac{d}{dx}(e^{nx}) \\ &= Am \cdot e^{mx} \cdot \frac{d}{dx}(mx) + Bn \cdot e^{nx} \cdot \frac{d}{dx}(nx) = Am^2e^{mx} + Bn^2e^{nx} \end{aligned}$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} - (m+n) \frac{dy}{dx} + mny &= Am^2e^{mx} + Bn^2e^{nx} - (m+n) \cdot (Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{nx}) \\ &= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} - Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx} \\ &= 0 \end{aligned}$$

Hence, proved.

Question 15:

If $y = 500e^{7x} + 600e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49y$

Answer

It is given that, $y = 500e^{7x} + 600e^{-7x}$

Then,

$$\begin{aligned}
 \frac{dy}{dx} &= 500 \cdot \frac{d}{dx}(e^{7x}) + 600 \cdot \frac{d}{dx}(e^{-7x}) \\
 &= 500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) + 600 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x) \\
 &= 3500e^{7x} - 4200e^{-7x} \\
 \therefore \frac{d^2y}{dx^2} &= 3500 \cdot \frac{d}{dx}(e^{7x}) - 4200 \cdot \frac{d}{dx}(e^{-7x}) \\
 &= 3500 \cdot e^{7x} \cdot \frac{d}{dx}(7x) - 4200 \cdot e^{-7x} \cdot \frac{d}{dx}(-7x) \\
 &= 7 \times 3500 \cdot e^{7x} + 7 \times 4200 \cdot e^{-7x} \\
 &= 49 \times 500e^{7x} + 49 \times 600e^{-7x} \\
 &= 49(500e^{7x} + 600e^{-7x}) \\
 &= 49y
 \end{aligned}$$

Hence, proved.

Question 16:

If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Answer

The given relationship is $e^y(x+1) = 1$

$$e^y(x+1) = 1$$

$$\Rightarrow e^y = \frac{1}{x+1}$$

Taking logarithm on both the sides, we obtain

$$y = \log \frac{1}{(x+1)}$$

Differentiating this relationship with respect to x , we obtain

$$\frac{dy}{dx} = (x+1) \frac{d}{dx} \left(\frac{1}{x+1} \right) = (x+1) \cdot \frac{-1}{(x+1)^2} = \frac{-1}{x+1}$$

$$\therefore \frac{d^2y}{dx^2} = - \frac{d}{dx} \left(\frac{1}{x+1} \right) = - \left(\frac{-1}{(x+1)^2} \right) = \frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{-1}{x+1} \right)^2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx} \right)^2$$

Hence, proved.

Question 17:

If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1 = 2$

Answer

The given relationship is $y = (\tan^{-1} x)^2$

Then,

$$y_1 = 2 \tan^{-1} x \frac{d}{dx} (\tan^{-1} x)$$

$$\Rightarrow y_1 = 2 \tan^{-1} x \cdot \frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)y_1 = 2 \tan^{-1} x$$

Again differentiating with respect to x on both the sides, we obtain

$$(1+x^2)y_2 + 2xy_1 = 2 \left(\frac{1}{1+x^2} \right)$$

$$\Rightarrow (1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$$

Hence, proved.