

Miscellaneous Solutions

Question 1:

$$(3x^2 - 9x + 5)^9$$

Answer

$$\text{Let } y = (3x^2 - 9x + 5)^9$$

Using chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(3x^2 - 9x + 5)^9 \\ &= 9(3x^2 - 9x + 5)^8 \cdot \frac{d}{dx}(3x^2 - 9x + 5) \\ &= 9(3x^2 - 9x + 5)^8 \cdot (6x - 9) \\ &= 9(3x^2 - 9x + 5)^8 \cdot 3(2x - 3) \\ &= 27(3x^2 - 9x + 5)^8 (2x - 3)\end{aligned}$$

Question 2:

$$\sin^3 x + \cos^6 x$$

Answer

$$\text{Let } y = \sin^3 x + \cos^6 x$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{d}{dx}(\sin^3 x) + \frac{d}{dx}(\cos^6 x) \\ &= 3\sin^2 x \cdot \frac{d}{dx}(\sin x) + 6\cos^5 x \cdot \frac{d}{dx}(\cos x) \\ &= 3\sin^2 x \cdot \cos x + 6\cos^5 x \cdot (-\sin x) \\ &= 3\sin x \cos x (\sin x - 2\cos^4 x)\end{aligned}$$

Question 3:

$$(5x)^{3\cos 2x}$$

Answer

$$\text{Let } y = (5x)^{3\cos 2x}$$

Taking logarithm on both the sides, we obtain

$$\log y = 3 \cos 2x \log 5x$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= 3 \left[\log 5x \cdot \frac{d}{dx}(\cos 2x) + \cos 2x \cdot \frac{d}{dx}(\log 5x) \right] \\ \Rightarrow \frac{dy}{dx} &= 3y \left[\log 5x (-\sin 2x) \cdot \frac{d}{dx}(2x) + \cos 2x \cdot \frac{1}{5x} \cdot \frac{d}{dx}(5x) \right] \\ \Rightarrow \frac{dy}{dx} &= 3y \left[-2 \sin 2x \log 5x + \frac{\cos 2x}{x} \right] \\ \Rightarrow \frac{dy}{dx} &= 3y \left[\frac{3 \cos 2x}{x} - 6 \sin 2x \log 5x \right] \\ \therefore \frac{dy}{dx} &= (5x)^{3\cos 2x} \left[\frac{3 \cos 2x}{x} - 6 \sin 2x \log 5x \right]\end{aligned}$$

Question 4:

$$\sin^{-1}(x\sqrt{x}), \quad 0 \leq x \leq 1$$

Answer

$$\text{Let } y = \sin^{-1}(x\sqrt{x})$$

Using chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \sin^{-1}(x\sqrt{x}) \\ &= \frac{1}{\sqrt{1-(x\sqrt{x})^2}} \times \frac{d}{dx}(x\sqrt{x}) \\ &= \frac{1}{\sqrt{1-x^3}} \cdot \frac{d}{dx}\left(x^{\frac{3}{2}}\right) \\ &= \frac{1}{\sqrt{1-x^3}} \times \frac{3}{2} \cdot x^{\frac{1}{2}} \\ &= \frac{3\sqrt{x}}{2\sqrt{1-x^3}} \\ &= \frac{3}{2} \sqrt{\frac{x}{1-x^3}}\end{aligned}$$

Question 5:

$$\frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}, \quad -2 < x < 2$$

Answer

$$\text{Let } y = \frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}$$

By quotient rule, we obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sqrt{2x+7} \frac{d}{dx} \left(\cos^{-1} \frac{x}{2} \right) - \left(\cos^{-1} \frac{x}{2} \right) \frac{d}{dx} (\sqrt{2x+7})}{(\sqrt{2x+7})^2} \\ &= \frac{\sqrt{2x+7} \left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{d}{dx} \left(\frac{x}{2} \right) \right] - \left(\cos^{-1} \frac{x}{2} \right) \frac{1}{2\sqrt{2x+7}} \cdot \frac{d}{dx} (2x+7)}{2x+7} \\ &= \frac{\sqrt{2x+7} \frac{-1}{\sqrt{4-x^2}} - \left(\cos^{-1} \frac{x}{2} \right) \frac{2}{2\sqrt{2x+7}}}{2x+7} \\ &= \frac{-\sqrt{2x+7}}{\sqrt{4-x^2} \times (2x+7)} - \frac{\cos^{-1} \frac{x}{2}}{(\sqrt{2x+7})(2x+7)} \\ &= - \left[\frac{1}{\sqrt{4-x^2} \sqrt{2x+7}} + \frac{\cos^{-1} \frac{x}{2}}{(2x+7)^{\frac{3}{2}}} \right] \end{aligned}$$

Question 6:

$$\cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right], 0 < x < \frac{\pi}{2}$$

Answer

$$\text{Let } y = \cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right] \quad \dots(1)$$

$$\begin{aligned} \text{Then, } & \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \\ &= \frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x} - \sqrt{1-\sin x})(\sqrt{1+\sin x} + \sqrt{1-\sin x})} \\ &= \frac{(1+\sin x) + (1-\sin x) + 2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x) - (1-\sin x)} \\ &= \frac{2 + 2\sqrt{1-\sin^2 x}}{2\sin x} \\ &= \frac{1 + \cos x}{\sin x} \\ &= \frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} \\ &= \cot \frac{x}{2} \end{aligned}$$

Therefore, equation (1) becomes

$$\begin{aligned} y &= \cot^{-1} \left(\cot \frac{x}{2} \right) \\ \Rightarrow y &= \frac{x}{2} \\ \therefore \frac{dy}{dx} &= \frac{1}{2} \frac{d}{dx} (x) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{2} \end{aligned}$$

Question 7:

$$(\log x)^{\log x}, \quad x > 1$$

Answer

$$\text{Let } y = (\log x)^{\log x}$$

Taking logarithm on both the sides, we obtain

$$\log y = \log x \cdot \log(\log x)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx} [\log x \cdot \log(\log x)] \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \log(\log x) \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx} [\log(\log x)] \\ \Rightarrow \frac{dy}{dx} &= y \left[\log(\log x) \cdot \frac{1}{x} + \log x \cdot \frac{1}{\log x} \cdot \frac{d}{dx}(\log x) \right] \\ \Rightarrow \frac{dy}{dx} &= y \left[\frac{1}{x} \log(\log x) + \frac{1}{x} \right] \\ \therefore \frac{dy}{dx} &= (\log x)^{\log x} \left[\frac{1}{x} + \frac{\log(\log x)}{x} \right] \end{aligned}$$

Question 8:

$$\cos(a \cos x + b \sin x), \text{ for some constant } a \text{ and } b.$$

Answer

$$\text{Let } y = \cos(a \cos x + b \sin x)$$

By using chain rule, we obtain

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \cos(a \cos x + b \sin x) \\ \Rightarrow \frac{dy}{dx} &= -\sin(a \cos x + b \sin x) \cdot \frac{d}{dx}(a \cos x + b \sin x) \\ &= -\sin(a \cos x + b \sin x) \cdot [a(-\sin x) + b \cos x] \\ &= (a \sin x - b \cos x) \cdot \sin(a \cos x + b \sin x) \end{aligned}$$

Question 9:

$$(\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$$

Answer

$$\text{Let } y = (\sin x - \cos x)^{(\sin x - \cos x)}$$

Taking logarithm on both the sides, we obtain

$$\log y = \log \left[(\sin x - \cos x)^{(\sin x - \cos x)} \right]$$

$$\Rightarrow \log y = (\sin x - \cos x) \cdot \log(\sin x - \cos x)$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \left[(\sin x - \cos x) \log(\sin x - \cos x) \right]$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log(\sin x - \cos x) \cdot \frac{d}{dx} (\sin x - \cos x) + (\sin x - \cos x) \cdot \frac{d}{dx} \log(\sin x - \cos x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \log(\sin x - \cos x) \cdot (\cos x + \sin x) + (\sin x - \cos x) \cdot \frac{1}{(\sin x - \cos x)} \cdot \frac{d}{dx} (\sin x - \cos x)$$

$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} \left[(\cos x + \sin x) \cdot \log(\sin x - \cos x) + (\cos x + \sin x) \right]$$

$$\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} (\cos x + \sin x) [1 + \log(\sin x - \cos x)]$$

Question 10:

$$x^x + x^a + a^x + a^a, \text{ for some fixed } a > 0 \text{ and } x > 0$$

Answer

$$\text{Let } y = x^x + x^a + a^x + a^a$$

$$\text{Also, let } x^x = u, x^a = v, a^x = w, \text{ and } a^a = s$$

$$\therefore y = u + v + w + s$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \quad \dots(1)$$

$$u = x^x$$

$$\Rightarrow \log u = \log x^x$$

$$\Rightarrow \log u = x \log x$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{u} \frac{du}{dx} &= \log x \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log x) \\ \Rightarrow \frac{du}{dx} &= u \left[\log x \cdot 1 + x \cdot \frac{1}{x} \right] \\ \Rightarrow \frac{du}{dx} &= x^x [\log x + 1] = x^x (1 + \log x) \quad \dots(2)\end{aligned}$$

$$\begin{aligned}v &= x^a \\ \therefore \frac{dv}{dx} &= \frac{d}{dx}(x^a) \\ \Rightarrow \frac{dv}{dx} &= ax^{a-1} \quad \dots(3)\end{aligned}$$

$$\begin{aligned}w &= a^x \\ \Rightarrow \log w &= \log a^x \\ \Rightarrow \log w &= x \log a\end{aligned}$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned}\frac{1}{w} \cdot \frac{dw}{dx} &= \log a \cdot \frac{d}{dx}(x) \\ \Rightarrow \frac{dw}{dx} &= w \log a \\ \Rightarrow \frac{dw}{dx} &= a^x \log a \quad \dots(4)\end{aligned}$$

$$s = a^a$$

Since a is constant, a^a is also a constant.

$$\therefore \frac{ds}{dx} = 0 \quad \dots(5)$$

From (1), (2), (3), (4), and (5), we obtain

$$\begin{aligned}\frac{dy}{dx} &= x^x (1 + \log x) + ax^{a-1} + a^x \log a + 0 \\ &= x^x (1 + \log x) + ax^{a-1} + a^x \log a\end{aligned}$$

Question 11:

$$x^{x^2-3} + (x-3)^{x^2}, \text{ for } x > 3$$

Answer

$$\text{Let } y = x^{x^2-3} + (x-3)^{x^2}$$

$$\text{Also, let } u = x^{x^2-3} \text{ and } v = (x-3)^{x^2}$$

$$\therefore y = u + v$$

Differentiating both sides with respect to x , we obtain

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(1)$$

$$u = x^{x^2-3}$$

$$\therefore \log u = \log(x^{x^2-3})$$

$$\log u = (x^2 - 3) \log x$$

Differentiating with respect to x , we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \log x \cdot \frac{d}{dx}(x^2 - 3) + (x^2 - 3) \cdot \frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = \log x \cdot 2x + (x^2 - 3) \cdot \frac{1}{x}$$

$$\Rightarrow \frac{du}{dx} = x^{x^2-3} \cdot \left[\frac{x^2 - 3}{x} + 2x \log x \right]$$

Also,

$$v = (x-3)^{x^2}$$

$$\therefore \log v = \log(x-3)^{x^2}$$

$$\Rightarrow \log v = x^2 \log(x-3)$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{1}{v} \cdot \frac{dv}{dx} &= \log(x-3) \cdot \frac{d}{dx}(x^2) + x^2 \cdot \frac{d}{dx}[\log(x-3)] \\ \Rightarrow \frac{1}{v} \frac{dv}{dx} &= \log(x-3) \cdot 2x + x^2 \cdot \frac{1}{x-3} \cdot \frac{d}{dx}(x-3) \\ \Rightarrow \frac{dv}{dx} &= v \left[2x \log(x-3) + \frac{x^2}{x-3} \cdot 1 \right] \\ \Rightarrow \frac{dv}{dx} &= (x-3)^{x^2} \left[\frac{x^2}{x-3} + 2x \log(x-3) \right] \end{aligned}$$

Substituting the expressions of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx} = x^{x^2-3} \left[\frac{x^2-3}{x} + 2x \log x \right] + (x-3)^{x^2} \left[\frac{x^2}{x-3} + 2x \log(x-3) \right]$$

Question 12:

Find $\frac{dy}{dx}$, if $y = 12(1 - \cos t)$, $x = 10(t - \sin t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$

Answer

It is given that, $y = 12(1 - \cos t)$, $x = 10(t - \sin t)$

$$\therefore \frac{dx}{dt} = \frac{d}{dt}[10(t - \sin t)] = 10 \cdot \frac{d}{dt}(t - \sin t) = 10(1 - \cos t)$$

$$\frac{dy}{dt} = \frac{d}{dt}[12(1 - \cos t)] = 12 \cdot \frac{d}{dt}(1 - \cos t) = 12 \cdot [0 - (-\sin t)] = 12 \sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{12 \sin t}{10(1 - \cos t)} = \frac{12 \cdot 2 \sin \frac{t}{2} \cdot \cos \frac{t}{2}}{10 \cdot 2 \sin^2 \frac{t}{2}} = \frac{6}{5} \cot \frac{t}{2}$$

Question 13:

Find $\frac{dy}{dx}$, if $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$, $-1 \leq x \leq 1$

Answer

It is given that, $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d}{dx} \left[\sin^{-1} x + \sin^{-1} \sqrt{1-x^2} \right] \\ \Rightarrow \frac{dy}{dx} &= \frac{d}{dx} (\sin^{-1} x) + \frac{d}{dx} (\sin^{-1} \sqrt{1-x^2}) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{d}{dx} (\sqrt{1-x^2}) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{x} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx} (1-x^2) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{2x\sqrt{1-x^2}} (-2x) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} \\ \therefore \frac{dy}{dx} &= 0 \end{aligned}$$

Question 14:

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for, $-1 < x < 1$, prove that

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

Answer

It is given that,

$$x\sqrt{1+y} + y\sqrt{1+x} = 0$$

$$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$$

Squaring both sides, we obtain

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow x^2 + x^2y = y^2 + xy^2$$

$$\Rightarrow x^2 - y^2 = xy^2 - x^2y$$

$$\Rightarrow x^2 - y^2 = xy(y-x)$$

$$\Rightarrow (x+y)(x-y) = xy(y-x)$$

$$\therefore x+y = -xy$$

$$\Rightarrow (1+x)y = -x$$

$$\Rightarrow y = \frac{-x}{(1+x)}$$

Differentiating both sides with respect to x , we obtain

$$y = \frac{-x}{(1+x)}$$

$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2} = -\frac{(1+x) - x}{(1+x)^2} = -\frac{1}{(1+x)^2}$$

Hence, proved.

Question 15:

If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

is a constant independent of a and b .

Answer

It is given that, $(x-a)^2 + (y-b)^2 = c^2$

Differentiating both sides with respect to x , we obtain

$$\frac{d}{dx}[(x-a)^2] + \frac{d}{dx}[(y-b)^2] = \frac{d}{dx}(c^2)$$

$$\Rightarrow 2(x-a) \cdot \frac{d}{dx}(x-a) + 2(y-b) \cdot \frac{d}{dx}(y-b) = 0$$

$$\Rightarrow 2(x-a) \cdot 1 + 2(y-b) \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x-a)}{y-b} \quad \dots(1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{-(x-a)}{y-b} \right]$$

$$\begin{aligned}
&= - \left[\frac{(y-b) \cdot \frac{d}{dx}(x-a) - (x-a) \cdot \frac{d}{dx}(y-b)}{(y-b)^2} \right] \\
&= - \left[\frac{(y-b) - (x-a) \cdot \frac{dy}{dx}}{(y-b)^2} \right] \\
&= - \left[\frac{(y-b) - (x-a) \cdot \left\{ \frac{-(x-a)}{y-b} \right\}}{(y-b)^2} \right] \quad [\text{Using (1)}] \\
&= - \left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3} \right] \\
\therefore \left[\frac{1 + \left(\frac{dy}{dx} \right)^2}{\frac{d^2y}{dx^2}} \right]^{\frac{3}{2}} &= - \frac{\left[1 + \frac{(x-a)^2}{(y-b)^2} \right]^{\frac{3}{2}}}{\left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3} \right]} = - \frac{\left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^2} \right]^{\frac{3}{2}}}{\left[\frac{(y-b)^2 + (x-a)^2}{(y-b)^3} \right]} \\
&= \frac{\left[\frac{c^2}{(y-b)^2} \right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}} = \frac{c^3}{-\frac{c^2}{(y-b)^3}} \\
&= -c, \text{ which is constant and is independent of } a \text{ and } b
\end{aligned}$$

Hence, proved.

Question 16:

If $\cos y = x \cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$

Answer

It is given that, $\cos y = x \cos(a + y)$

$$\therefore \frac{d}{dx}[\cos y] = \frac{d}{dx}[x \cos(a + y)]$$

$$\Rightarrow -\sin y \frac{dy}{dx} = \cos(a + y) \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}[\cos(a + y)]$$

$$\Rightarrow -\sin y \frac{dy}{dx} = \cos(a + y) + x \cdot [-\sin(a + y)] \frac{dy}{dx}$$

$$\Rightarrow [x \sin(a + y) - \sin y] \frac{dy}{dx} = \cos(a + y) \quad \dots(1)$$

Since $\cos y = x \cos(a + y)$, $x = \frac{\cos y}{\cos(a + y)}$

Then, equation (1) reduces to

$$\left[\frac{\cos y}{\cos(a + y)} \cdot \sin(a + y) - \sin y \right] \frac{dy}{dx} = \cos(a + y)$$

$$\Rightarrow [\cos y \cdot \sin(a + y) - \sin y \cdot \cos(a + y)] \cdot \frac{dy}{dx} = \cos^2(a + y)$$

$$\Rightarrow \sin(a + y - y) \frac{dy}{dx} = \cos^2(a + y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a}$$

Hence, proved.

Question 17:

If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, find $\frac{d^2 y}{dx^2}$

Answer

It is given that, $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$

$$\begin{aligned}\therefore \frac{dx}{dt} &= a \cdot \frac{d}{dt}(\cos t + t \sin t) \\ &= a \left[-\sin t + \sin t \cdot \frac{d}{dx}(t) + t \cdot \frac{d}{dt}(\sin t) \right] \\ &= a[-\sin t + \sin t + t \cos t] = at \cos t\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt} &= a \cdot \frac{d}{dt}(\sin t - t \cos t) \\ &= a \left[\cos t - \left\{ \cos t \cdot \frac{d}{dt}(t) + t \cdot \frac{d}{dt}(\cos t) \right\} \right] \\ &= a[\cos t - \{\cos t - t \sin t\}] = at \sin t\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at \sin t}{at \cos t} = \tan t$$

$$\begin{aligned}\text{Then, } \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx}(\tan t) = \sec^2 t \cdot \frac{dt}{dx} \\ &= \sec^2 t \cdot \frac{1}{at \cos t} \quad \left[\frac{dx}{dt} = at \cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at \cos t} \right] \\ &= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}\end{aligned}$$

Question 18:

If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x , and find it.

Answer

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

It is known that,

$$\text{Therefore, when } x \geq 0, f(x) = |x|^3 = x^3$$

$$\text{In this case, } f'(x) = 3x^2 \text{ and hence, } f''(x) = 6x$$

$$\text{When } x < 0, f(x) = |x|^3 = (-x)^3 = -x^3$$

In this case, $f'(x) = -3x^2$ and hence, $f''(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$ exists for all real x and is given by,

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

Question 19:

Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n .

Answer

To prove: $P(n) : \frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n

For $n = 1$,

$$P(1) : \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$$

$\therefore P(n)$ is true for $n = 1$

Let $P(k)$ is true for some positive integer k .

$$P(k) : \frac{d}{dx}(x^k) = kx^{k-1}$$

That is,

It has to be proved that $P(k + 1)$ is also true.

$$\text{Consider } \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k)$$

$$= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k) \quad [\text{By applying product rule}]$$

$$= x^k \cdot 1 + x \cdot k \cdot x^{k-1}$$

$$= x^k + kx^k$$

$$= (k+1) \cdot x^k$$

$$= (k+1) \cdot x^{(k+1)-1}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Therefore, by the principle of mathematical induction, the statement $P(n)$ is true for every positive integer n .

Hence, proved.

Question 20:

Using the fact that $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines.

Answer

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Differentiating both sides with respect to x , we obtain

$$\begin{aligned} \frac{d}{dx} [\sin(A + B)] &= \frac{d}{dx} (\sin A \cos B) + \frac{d}{dx} (\cos A \sin B) \\ \Rightarrow \cos(A + B) \cdot \frac{d}{dx} (A + B) &= \cos B \cdot \frac{d}{dx} (\sin A) + \sin A \cdot \frac{d}{dx} (\cos B) \\ &\quad + \sin B \cdot \frac{d}{dx} (\cos A) + \cos A \cdot \frac{d}{dx} (\sin B) \\ \Rightarrow \cos(A + B) \cdot \frac{d}{dx} (A + B) &= \cos B \cdot \cos A \frac{dA}{dx} + \sin A (-\sin B) \frac{dB}{dx} \\ &\quad + \sin B (-\sin A) \frac{dA}{dx} + \cos A \cos B \frac{dB}{dx} \\ \Rightarrow \cos(A + B) \cdot \left[\frac{dA}{dx} + \frac{dB}{dx} \right] &= (\cos A \cos B - \sin A \sin B) \cdot \left[\frac{dA}{dx} + \frac{dB}{dx} \right] \\ \therefore \cos(A + B) &= \cos A \cos B - \sin A \sin B \end{aligned}$$

Question 22:

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}, \text{ prove that } \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

If

Answer

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\Rightarrow y = (mc - nb)f(x) - (lc - na)g(x) + (lb - ma)h(x)$$

$$\text{Then, } \frac{dy}{dx} = \frac{d}{dx}[(mc - nb)f(x)] - \frac{d}{dx}[(lc - na)g(x)] + \frac{d}{dx}[(lb - ma)h(x)]$$

$$= (mc - nb)f'(x) - (lc - na)g'(x) + (lb - ma)h'(x)$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Thus,

Question 23:

If $y = e^{a \cos^{-1} x}$, $-1 \leq x \leq 1$, show that $(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$

Answer

It is given that, $y = e^{a \cos^{-1} x}$

Taking logarithm on both the sides, we obtain

$$\log y = a \cos^{-1} x \log e$$

$$\log y = a \cos^{-1} x$$

Differentiating both sides with respect to x , we obtain

$$\frac{1}{y} \frac{dy}{dx} = a \times \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2 = \frac{a^2 y^2}{1-x^2}$$

$$\Rightarrow (1-x^2) \left(\frac{dy}{dx}\right)^2 = a^2 y^2$$

$$(1-x^2) \left(\frac{dy}{dx}\right)^2 = a^2 y^2$$

Again differentiating both sides with respect to x , we obtain

$$\left(\frac{dy}{dx}\right)^2 \frac{d}{dx}(1-x^2) + (1-x^2) \times \frac{d}{dx} \left[\left(\frac{dy}{dx}\right)^2 \right] = a^2 \frac{d}{dx}(y^2)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + (1-x^2) \times 2 \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} = a^2 \cdot 2y \cdot \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 (-2x) + (1-x^2) \times 2 \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} = a^2 \cdot 2y \cdot \frac{dy}{dx}$$

$$\Rightarrow -x \frac{dy}{dx} + (1-x^2) \frac{d^2 y}{dx^2} = a^2 \cdot y \quad \left[\frac{dy}{dx} \neq 0 \right]$$

$$\Rightarrow (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$$

Hence, proved.