

## Exercise 6.2

**Question 1:**

Show that the function given by  $f(x) = 3x + 17$  is strictly increasing on  $\mathbf{R}$ .

Answer

Let  $x_1$  and  $x_2$  be any two numbers in  $\mathbf{R}$ .

Then, we have:

$$x_1 < x_2 \Rightarrow 3x_1 < 3x_2 \Rightarrow 3x_1 + 17 < 3x_2 + 17 \Rightarrow f(x_1) < f(x_2)$$

Hence,  $f$  is strictly increasing on  $\mathbf{R}$ .

**Alternate method:**

$f'(x) = 3 > 0$ , in every interval of  $\mathbf{R}$ .

Thus, the function is strictly increasing on  $\mathbf{R}$ .

**Question 2:**

Show that the function given by  $f(x) = e^{2x}$  is strictly increasing on  $\mathbf{R}$ .

Answer

Let  $x_1$  and  $x_2$  be any two numbers in  $\mathbf{R}$ .

Then, we have:

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow e^{2x_1} < e^{2x_2} \Rightarrow f(x_1) < f(x_2)$$

Hence,  $f$  is strictly increasing on  $\mathbf{R}$ .

**Question 3:**

Show that the function given by  $f(x) = \sin x$  is

(a) strictly increasing in  $\left(0, \frac{\pi}{2}\right)$  (b) strictly decreasing in  $\left(\frac{\pi}{2}, \pi\right)$

(c) neither increasing nor decreasing in  $(0, \pi)$

Answer

The given function is  $f(x) = \sin x$ .

$$\therefore f'(x) = \cos x$$

(a) Since for each  $x \in \left(0, \frac{\pi}{2}\right)$ ,  $\cos x > 0$ , we have  $f'(x) > 0$ .

Hence,  $f$  is strictly increasing in  $\left(0, \frac{\pi}{2}\right)$ .

(b) Since for each  $x \in \left(\frac{\pi}{2}, \pi\right)$ ,  $\cos x < 0$ , we have  $f'(x) < 0$ .

Hence,  $f$  is strictly decreasing in  $\left(\frac{\pi}{2}, \pi\right)$ .

(c) From the results obtained in (a) and (b), it is clear that  $f$  is neither increasing nor decreasing in  $(0, \pi)$ .

#### Question 4:

Find the intervals in which the function  $f$  given by  $f(x) = 2x^2 - 3x$  is

(a) strictly increasing (b) strictly decreasing

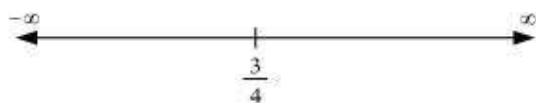
Answer

The given function is  $f(x) = 2x^2 - 3x$ .

$$f'(x) = 4x - 3$$

$$\therefore f'(x) = 0 \Rightarrow x = \frac{3}{4}$$

Now, the point  $\frac{3}{4}$  divides the real line into two disjoint intervals i.e.,  $\left(-\infty, \frac{3}{4}\right)$  and  $\left(\frac{3}{4}, \infty\right)$ .



In interval  $\left(-\infty, \frac{3}{4}\right)$ ,  $f'(x) = 4x - 3 < 0$ .

Hence, the given function ( $f$ ) is strictly decreasing in interval  $\left(-\infty, \frac{3}{4}\right)$ .

In interval  $\left(\frac{3}{4}, \infty\right)$ ,  $f'(x) = 4x - 3 > 0$ .

Hence, the given function ( $f$ ) is strictly increasing in interval  $\left(\frac{3}{4}, \infty\right)$ .

**Question 5:**

Find the intervals in which the function  $f$  given by  $f(x) = 2x^3 - 3x^2 - 36x + 7$  is

(a) strictly increasing (b) strictly decreasing

Answer

The given function is  $f(x) = 2x^3 - 3x^2 - 36x + 7$ .

$$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x+2)(x-3)$$

$$\therefore f'(x) = 0 \Rightarrow x = -2, 3$$

The points  $x = -2$  and  $x = 3$  divide the real line into three disjoint intervals i.e.,

$(-\infty, -2)$ ,  $(-2, 3)$ , and  $(3, \infty)$ .



In intervals  $(-\infty, -2)$  and  $(3, \infty)$ ,  $f'(x)$  is positive while in interval

$(-2, 3)$ ,  $f'(x)$  is negative.

Hence, the given function ( $f$ ) is strictly increasing in intervals

$(-\infty, -2)$  and  $(3, \infty)$ , while function ( $f$ ) is strictly decreasing in interval

$(-2, 3)$ .

**Question 6:**

Find the intervals in which the following functions are strictly increasing or decreasing:

(a)  $x^2 + 2x - 5$  (b)  $10 - 6x - 2x^2$

(c)  $-2x^3 - 9x^2 - 12x + 1$  (d)  $6 - 9x - x^2$

(e)  $(x + 1)^3 (x - 3)^3$

Answer

(a) We have,

$$f(x) = x^2 + 2x - 5$$

$$\therefore f'(x) = 2x + 2$$

Now,

$$f'(x) = 0 \Rightarrow x = -1$$

Point  $x = -1$  divides the real line into two disjoint intervals i.e.,  $(-\infty, -1)$  and  $(-1, \infty)$ .

In interval  $(-\infty, -1)$ ,  $f'(x) = 2x + 2 < 0$ .

$\therefore f$  is strictly decreasing in interval  $(-\infty, -1)$ .

Thus,  $f$  is strictly decreasing for  $x < -1$ .

In interval  $(-1, \infty)$ ,  $f'(x) = 2x + 2 > 0$ .

$\therefore f$  is strictly increasing in interval  $(-1, \infty)$ .

Thus,  $f$  is strictly increasing for  $x > -1$ .

(b) We have,

$$f(x) = 10 - 6x - 2x^2$$

$$\therefore f'(x) = -6 - 4x$$

Now,

$$f'(x) = 0 \Rightarrow x = -\frac{3}{2}$$

The point  $x = -\frac{3}{2}$  divides the real line into two disjoint intervals

i.e.,  $\left(-\infty, -\frac{3}{2}\right)$  and  $\left(-\frac{3}{2}, \infty\right)$ .

In interval  $\left(-\infty, -\frac{3}{2}\right)$  i.e., when  $x < -\frac{3}{2}$ ,  $f'(x) = -6 - 4x < 0$ .

$\therefore f$  is strictly increasing for  $x < -\frac{3}{2}$ .

In interval  $\left(-\frac{3}{2}, \infty\right)$  i.e., when  $x > -\frac{3}{2}$ ,  $f'(x) = -6 - 4x < 0$ .

$\therefore f$  is strictly decreasing for  $x > -\frac{3}{2}$ .

(c) We have,

$$f(x) = -2x^3 - 9x^2 - 12x + 1$$

$$\therefore f'(x) = -6x^2 - 18x - 12 = -6(x^2 + 3x + 2) = -6(x+1)(x+2)$$

Now,

$$f'(x) = 0 \Rightarrow x = -1 \text{ and } x = -2$$

Points  $x = -1$  and  $x = -2$  divide the real line into three disjoint intervals

i.e.,  $(-\infty, -2)$ ,  $(-2, -1)$ , and  $(-1, \infty)$ .

In intervals  $(-\infty, -2)$  and  $(-1, \infty)$  i.e., when  $x < -2$  and  $x > -1$ ,

$$f'(x) = -6(x+1)(x+2) < 0$$

$\therefore f$  is strictly decreasing for  $x < -2$  and  $x > -1$ .

Now, in interval  $(-2, -1)$  i.e., when  $-2 < x < -1$ ,  $f'(x) = -6(x+1)(x+2) > 0$ .

$\therefore f$  is strictly increasing for  $-2 < x < -1$ .

(d) We have,

$$f(x) = 6 - 9x - x^2$$

$$\therefore f'(x) = -9 - 2x$$

Now,  $f'$

$$(x) = 0 \text{ gives } x = -\frac{9}{2}$$

The point  $x = -\frac{9}{2}$  divides the real line into two disjoint intervals i.e.,

$$\left(-\infty, -\frac{9}{2}\right) \text{ and } \left(-\frac{9}{2}, \infty\right)$$

In interval  $\left(-\infty, -\frac{9}{2}\right)$  i.e., for  $x < -\frac{9}{2}$ ,  $f'(x) = -9 - 2x > 0$ .

$\therefore f$  is strictly increasing for  $x < -\frac{9}{2}$ .

In interval  $\left(-\frac{9}{2}, \infty\right)$  i.e., for  $x > -\frac{9}{2}$ ,  $f'(x) = -9 - 2x < 0$ .

$\therefore f$  is strictly decreasing for  $x > -\frac{9}{2}$ .

(e) We have,

$$f(x) = (x + 1)^3 (x - 3)^3$$

$$\begin{aligned} f'(x) &= 3(x+1)^2(x-3)^3 + 3(x-3)^2(x+1)^3 \\ &= 3(x+1)^2(x-3)^2[x-3+x+1] \\ &= 3(x+1)^2(x-3)^2(2x-2) \\ &= 6(x+1)^2(x-3)^2(x-1) \end{aligned}$$

Now,

$$f'(x) = 0 \Rightarrow x = -1, 3, 1$$

The points  $x = -1$ ,  $x = 1$ , and  $x = 3$  divide the real line into four disjoint intervals

i.e.,  $(-\infty, -1)$ ,  $(-1, 1)$ ,  $(1, 3)$ , and  $(3, \infty)$ .

In intervals  $(-\infty, -1)$  and  $(-1, 1)$ ,  $f'(x) = 6(x+1)^2(x-3)^2(x-1) < 0$ .

$\therefore f$  is strictly decreasing in intervals  $(-\infty, -1)$  and  $(-1, 1)$ .

In intervals  $(1, 3)$  and  $(3, \infty)$ ,  $f'(x) = 6(x+1)^2(x-3)^2(x-1) > 0$ .

$\therefore f$  is strictly increasing in intervals  $(1, 3)$  and  $(3, \infty)$ .

**Question 7:**

Show that  $y = \log(1+x) - \frac{2x}{2+x}, x > -1$ , is an increasing function of  $x$  throughout its domain.

Answer

We have,

$$y = \log(1+x) - \frac{2x}{2+x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x} - \frac{(2+x)(2) - 2x(1)}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2} = \frac{x^2}{(2+x)^2}$$

$$\text{Now, } \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{x^2}{(2+x)^2} = 0$$

$$\Rightarrow x^2 = 0 \quad [(2+x) \neq 0 \text{ as } x > -1]$$

$$\Rightarrow x = 0$$

Since  $x > -1$ , point  $x = 0$  divides the domain  $(-1, \infty)$  in two disjoint intervals i.e.,  $-1 < x < 0$  and  $x > 0$ .

When  $-1 < x < 0$ , we have:

$$x < 0 \Rightarrow x^2 > 0$$

$$x > -1 \Rightarrow (2+x) > 0 \Rightarrow (2+x)^2 > 0$$

$$\therefore y' = \frac{x^2}{(2+x)^2} > 0$$

Also, when  $x > 0$ :

$$x > 0 \Rightarrow x^2 > 0, (2+x)^2 > 0$$

$$\therefore y' = \frac{x^2}{(2+x)^2} > 0$$

Hence, function  $f$  is increasing throughout this domain.

**Question 8:**



Find the values of  $x$  for which  $y = [x(x-2)]^2$  is an increasing function.

Answer

We have,

$$y = [x(x-2)]^2 = [x^2 - 2x]^2$$

$$\therefore \frac{dy}{dx} = y' = 2(x^2 - 2x)(2x - 2) = 4x(x-2)(x-1)$$

$$\therefore \frac{dy}{dx} = 0 \Rightarrow x = 0, x = 2, x = 1.$$

The points  $x = 0$ ,  $x = 1$ , and  $x = 2$  divide the real line into four disjoint intervals i.e.,  $(-\infty, 0)$ ,  $(0, 1)$ ,  $(1, 2)$ , and  $(2, \infty)$ .

In intervals  $(-\infty, 0)$  and  $(1, 2)$ ,  $\frac{dy}{dx} < 0$ .

$\therefore y$  is strictly decreasing in intervals  $(-\infty, 0)$  and  $(1, 2)$ .

However, in intervals  $(0, 1)$  and  $(2, \infty)$ ,  $\frac{dy}{dx} > 0$ .

$\therefore y$  is strictly increasing in intervals  $(0, 1)$  and  $(2, \infty)$ .

$\therefore y$  is strictly increasing for  $0 < x < 1$  and  $x > 2$ .

**Question 9:**

Prove that  $y = \frac{4 \sin \theta}{(2 + \cos \theta)} - \theta$  is an increasing function of  $\theta$  in  $\left[0, \frac{\pi}{2}\right]$ .

Answer

We have,

$$y = \frac{4 \sin \theta}{(2 + \cos \theta)} - \theta$$

$$\therefore \frac{dy}{dx} = \frac{(2 + \cos \theta)(4 \cos \theta) - 4 \sin \theta(-\sin \theta)}{(2 + \cos \theta)^2} - 1$$

$$= \frac{8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta}{(2 + \cos \theta)^2} - 1$$

$$= \frac{8 \cos \theta + 4}{(2 + \cos \theta)^2} - 1$$

Now,  $\frac{dy}{dx} = 0$ .

$$\Rightarrow \frac{8 \cos \theta + 4}{(2 + \cos \theta)^2} = 1$$

$$\Rightarrow 8 \cos \theta + 4 = 4 + \cos^2 \theta + 4 \cos \theta$$

$$\Rightarrow \cos^2 \theta - 4 \cos \theta = 0$$

$$\Rightarrow \cos \theta (\cos \theta - 4) = 0$$

$$\Rightarrow \cos \theta = 0 \text{ or } \cos \theta = 4$$

Since  $\cos \theta \neq 4$ ,  $\cos \theta = 0$ .

$$\cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

Now,

$$\frac{dy}{dx} = \frac{8 \cos \theta + 4 - (4 + \cos^2 \theta + 4 \cos \theta)}{(2 + \cos \theta)^2} = \frac{4 \cos \theta - \cos^2 \theta}{(2 + \cos \theta)^2} = \frac{\cos \theta (4 - \cos \theta)}{(2 + \cos \theta)^2}$$

In interval  $\left(0, \frac{\pi}{2}\right)$ , we have  $\cos \theta > 0$ . Also,  $4 > \cos \theta \Rightarrow 4 - \cos \theta > 0$ .

$$\therefore \cos \theta (4 - \cos \theta) > 0 \text{ and also } (2 + \cos \theta)^2 > 0$$

$$\Rightarrow \frac{\cos \theta (4 - \cos \theta)}{(2 + \cos \theta)^2} > 0$$

$$\Rightarrow \frac{dy}{dx} > 0$$

Therefore,  $y$  is strictly increasing in interval  $\left(0, \frac{\pi}{2}\right)$ .

Also, the given function is continuous at  $x = 0$  and  $x = \frac{\pi}{2}$ .

Hence,  $y$  is increasing in interval  $\left[0, \frac{\pi}{2}\right]$ .

**Question 10:**

Prove that the logarithmic function is strictly increasing on  $(0, \infty)$ .

Answer

The given function is  $f(x) = \log x$ .

$$\therefore f'(x) = \frac{1}{x}$$

It is clear that for  $x > 0$ ,  $f'(x) = \frac{1}{x} > 0$ .

Hence,  $f(x) = \log x$  is strictly increasing in interval  $(0, \infty)$ .

**Question 11:**

Prove that the function  $f$  given by  $f(x) = x^2 - x + 1$  is neither strictly increasing nor strictly decreasing on  $(-1, 1)$ .

Answer

The given function is  $f(x) = x^2 - x + 1$ .

$$\therefore f'(x) = 2x - 1$$

$$\text{Now, } f'(x) = 0 \Rightarrow x = \frac{1}{2}.$$

The point  $\frac{1}{2}$  divides the interval  $(-1, 1)$  into two disjoint intervals

$$\text{i.e., } \left(-1, \frac{1}{2}\right) \text{ and } \left(\frac{1}{2}, 1\right).$$

Now, in interval  $\left(-1, \frac{1}{2}\right)$ ,  $f'(x) = 2x - 1 < 0$ .

Therefore,  $f$  is strictly decreasing in interval  $\left(-1, \frac{1}{2}\right)$ .

However, in interval  $\left(\frac{1}{2}, 1\right)$ ,  $f'(x) = 2x - 1 > 0$ .

Therefore,  $f$  is strictly increasing in interval  $\left(\frac{1}{2}, 1\right)$ .

Hence,  $f$  is neither strictly increasing nor decreasing in interval  $(-1, 1)$ .

#### Question 12:

Which of the following functions are strictly decreasing on  $\left(0, \frac{\pi}{2}\right)$ ?

(A)  $\cos x$  (B)  $\cos 2x$  (C)  $\cos 3x$  (D)  $\tan x$

Answer

$$\text{(A) Let } f_1(x) = \cos x.$$

$$\therefore f_1'(x) = -\sin x$$

In interval  $\left(0, \frac{\pi}{2}\right)$ ,  $f_1'(x) = -\sin x < 0$ .

$\therefore f_1(x) = \cos x$  is strictly decreasing in interval  $\left(0, \frac{\pi}{2}\right)$ .

(B) Let  $f_2(x) = \cos 2x$ .

$$\therefore f_2'(x) = -2 \sin 2x$$

$$\text{Now, } 0 < x < \frac{\pi}{2} \Rightarrow 0 < 2x < \pi \Rightarrow \sin 2x > 0 \Rightarrow -2 \sin 2x < 0$$

$$\therefore f_2'(x) = -2 \sin 2x < 0 \text{ on } \left(0, \frac{\pi}{2}\right)$$

$\therefore f_2(x) = \cos 2x$  is strictly decreasing in interval  $\left(0, \frac{\pi}{2}\right)$ .

(C) Let  $f_3(x) = \cos 3x$ .

$$\therefore f_3'(x) = -3 \sin 3x$$

$$\text{Now, } f_3'(x) = 0.$$

$$\Rightarrow \sin 3x = 0 \Rightarrow 3x = \pi, \text{ as } x \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow x = \frac{\pi}{3}$$

The point  $x = \frac{\pi}{3}$  divides the interval  $\left(0, \frac{\pi}{2}\right)$  into two disjoint intervals

i.e.,  $\left(0, \frac{\pi}{3}\right)$  and  $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ .

Now, in interval  $\left(0, \frac{\pi}{3}\right)$ ,  $f_3'(x) = -3 \sin 3x < 0$  [as  $0 < x < \frac{\pi}{3} \Rightarrow 0 < 3x < \pi$ ].

$\therefore f_3$  is strictly decreasing in interval  $\left(0, \frac{\pi}{3}\right)$ .

However, in interval  $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ ,  $f_3'(x) = -3 \sin 3x > 0$  [as  $\frac{\pi}{3} < x < \frac{\pi}{2} \Rightarrow \pi < 3x < \frac{3\pi}{2}$ ].

$\therefore f_3$  is strictly increasing in interval  $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ .

Hence,  $f_3$  is neither increasing nor decreasing in interval  $\left(0, \frac{\pi}{2}\right)$ .

(D) Let  $f_4(x) = \tan x$ .

$$\therefore f_4'(x) = \sec^2 x$$

In interval  $\left(0, \frac{\pi}{2}\right), f_4'(x) = \sec^2 x > 0$ .

$\therefore f_4$  is strictly increasing in interval  $\left(0, \frac{\pi}{2}\right)$ .

Therefore, functions  $\cos x$  and  $\cos 2x$  are strictly decreasing in  $\left(0, \frac{\pi}{2}\right)$ .  
Hence, the correct answers are A and B.

### Question 13:

On which of the following intervals is the function  $f$  given by  $f(x) = x^{100} + \sin x - 1$  strictly decreasing?

(A)  $(0, 1)$  (B)  $\left(\frac{\pi}{2}, \pi\right)$

(C)  $\left(0, \frac{\pi}{2}\right)$  (D) None of these

Answer

We have,

$$f(x) = x^{100} + \sin x - 1$$

$$\therefore f'(x) = 100x^{99} + \cos x$$

In interval  $(0, 1)$ ,  $\cos x > 0$  and  $100x^{99} > 0$ .

$$\therefore f'(x) > 0.$$

Thus, function  $f$  is strictly increasing in interval  $(0, 1)$ .

In interval  $\left(\frac{\pi}{2}, \pi\right)$ ,  $\cos x < 0$  and  $100x^{99} > 0$ . Also,  $100x^{99} > \cos x$

$$\therefore f'(x) > 0 \text{ in } \left(\frac{\pi}{2}, \pi\right).$$

Thus, function  $f$  is strictly increasing in interval  $\left(\frac{\pi}{2}, \pi\right)$ .

In interval  $\left(0, \frac{\pi}{2}\right)$ ,  $\cos x > 0$  and  $100x^{99} > 0$ .

$$\therefore 100x^{99} + \cos x > 0$$

$$\Rightarrow f'(x) > 0 \text{ on } \left(0, \frac{\pi}{2}\right)$$

$\therefore f$  is strictly increasing in interval  $\left(0, \frac{\pi}{2}\right)$ .

Hence, function  $f$  is strictly decreasing in none of the intervals.

The correct answer is D.

#### Question 14:

Find the least value of  $a$  such that the function  $f$  given  $f(x) = x^2 + ax + 1$  is strictly increasing on  $(1, 2)$ .

Answer

We have,

$$f(x) = x^2 + ax + 1$$

$$\therefore f'(x) = 2x + a$$

Now, function  $f$  will be increasing in  $(1, 2)$ , if  $f'(x) > 0$  in  $(1, 2)$ .

$$f'(x) > 0$$

$$\Rightarrow 2x + a > 0$$

$$\Rightarrow 2x > -a$$

$$\Rightarrow x > \frac{-a}{2}$$

Therefore, we have to find the least value of  $a$  such that

$$x > \frac{-a}{2}, \text{ when } x \in (1, 2).$$

$$\Rightarrow x > \frac{-a}{2} \text{ (when } 1 < x < 2)$$

Thus, the least value of  $a$  for  $f$  to be increasing on  $(1, 2)$  is given by,

$$\frac{-a}{2} = 1$$

$$\frac{-a}{2} = 1 \Rightarrow a = -2$$

Hence, the required value of  $a$  is  $-2$ .

**Question 15:**



Let  $\mathbf{I}$  be any interval disjoint from  $(-1, 1)$ . Prove that the function  $f$  given by

$f(x) = x + \frac{1}{x}$  is strictly increasing on  $\mathbf{I}$ .

Answer

We have,

$$f(x) = x + \frac{1}{x}$$

$$\therefore f'(x) = 1 - \frac{1}{x^2}$$

Now,

$$f'(x) = 0 \Rightarrow \frac{1}{x^2} = 1 \Rightarrow x = \pm 1$$

The points  $x = 1$  and  $x = -1$  divide the real line in three disjoint intervals i.e.,

$(-\infty, -1)$ ,  $(-1, 1)$ , and  $(1, \infty)$ .

In interval  $(-1, 1)$ , it is observed that:

$$-1 < x < 1$$

$$\Rightarrow x^2 < 1$$

$$\Rightarrow 1 < \frac{1}{x^2}, x \neq 0$$

$$\Rightarrow 1 - \frac{1}{x^2} < 0, x \neq 0$$

$$\therefore f'(x) = 1 - \frac{1}{x^2} < 0 \text{ on } (-1, 1) \sim \{0\}.$$

$\therefore f$  is strictly decreasing on  $(-1, 1) \sim \{0\}$ .

In intervals  $(-\infty, -1)$  and  $(1, \infty)$ , it is observed that:

$$x < -1 \text{ or } 1 < x$$

$$\Rightarrow x^2 > 1$$

$$\Rightarrow 1 > \frac{1}{x^2}$$

$$\Rightarrow 1 - \frac{1}{x^2} > 0$$

$$\therefore f'(x) = 1 - \frac{1}{x^2} > 0 \text{ on } (-\infty, -1) \text{ and } (1, \infty).$$

$\therefore f$  is strictly increasing on  $(-\infty, -1)$  and  $(1, \infty)$ .

Hence, function  $f$  is strictly increasing in interval  $\mathbf{I}$  disjoint from  $(-1, 1)$ .

Hence, the given result is proved.

**Question 16:**

Prove that the function  $f$  given by  $f(x) = \log \sin x$  is strictly increasing on  $\left(0, \frac{\pi}{2}\right)$  and

strictly decreasing on  $\left(\frac{\pi}{2}, \pi\right)$ .

Answer

We have,

$$f(x) = \log \sin x$$

$$\therefore f'(x) = \frac{1}{\sin x} \cos x = \cot x$$

In interval  $\left(0, \frac{\pi}{2}\right), f'(x) = \cot x > 0$ .

$\therefore f$  is strictly increasing in  $\left(0, \frac{\pi}{2}\right)$ .

In interval  $\left(\frac{\pi}{2}, \pi\right)$ ,  $f'(x) = \cot x < 0$ .

$\therefore f$  is strictly decreasing in  $\left(\frac{\pi}{2}, \pi\right)$ .

**Question 17:**

Prove that the function  $f$  given by  $f(x) = \log \cos x$  is strictly decreasing on  $\left(0, \frac{\pi}{2}\right)$  and strictly increasing on  $\left(\frac{\pi}{2}, \pi\right)$ .

Answer

We have,

$$f(x) = \log \cos x$$

$$\therefore f'(x) = \frac{1}{\cos x}(-\sin x) = -\tan x$$

In interval  $\left(0, \frac{\pi}{2}\right)$ ,  $\tan x > 0 \Rightarrow -\tan x < 0$ .

$$\therefore f'(x) < 0 \text{ on } \left(0, \frac{\pi}{2}\right)$$

$\therefore f$  is strictly decreasing on  $\left(0, \frac{\pi}{2}\right)$ .

In interval  $\left(\frac{\pi}{2}, \pi\right)$ ,  $\tan x < 0 \Rightarrow -\tan x > 0$ .

$$\therefore f'(x) > 0 \text{ on } \left(\frac{\pi}{2}, \pi\right)$$

$\therefore f$  is strictly increasing on  $\left(\frac{\pi}{2}, \pi\right)$ .

**Question 18:**

Prove that the function given by  $f(x) = x^3 - 3x^2 + 3x - 100$  is increasing in  $\mathbf{R}$ .

Answer

We have,

$$f(x) = x^3 - 3x^2 + 3x - 100$$

$$\begin{aligned} f'(x) &= 3x^2 - 6x + 3 \\ &= 3(x^2 - 2x + 1) \\ &= 3(x-1)^2 \end{aligned}$$

For any  $x \in \mathbf{R}$ ,  $(x - 1)^2 > 0$ .

Thus,  $f'(x)$  is always positive in  $\mathbf{R}$ .

Hence, the given function ( $f$ ) is increasing in  $\mathbf{R}$ .

**Question 19:**

The interval in which  $y = x^2 e^{-x}$  is increasing is

(A)  $(-\infty, \infty)$  (B)  $(-2, 0)$  (C)  $(2, \infty)$  (D)  $(0, 2)$

Answer

We have,

$$y = x^2 e^{-x}$$

$$\therefore \frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x)$$

$$\text{Now, } \frac{dy}{dx} = 0.$$

$$\Rightarrow x = 0 \text{ and } x = 2$$

The points  $x = 0$  and  $x = 2$  divide the real line into three disjoint intervals

i.e.,  $(-\infty, 0)$ ,  $(0, 2)$ , and  $(2, \infty)$ .

In intervals  $(-\infty, 0)$  and  $(2, \infty)$ ,  $f'(x) < 0$  as  $e^{-x}$  is always positive.

$\therefore f$  is decreasing on  $(-\infty, 0)$  and  $(2, \infty)$ .

In interval  $(0, 2)$ ,  $f'(x) > 0$ .

$\therefore f$  is strictly increasing on  $(0, 2)$ .

Hence,  $f$  is strictly increasing in interval  $(0, 2)$ .

The correct answer is D.