

## Miscellaneous Solutions

**Question 1:**

$$\frac{1}{x-x^3}$$

Answer

$$\frac{1}{x-x^3} = \frac{1}{x(1-x^2)} = \frac{1}{x(1-x)(1+x)}$$

$$\text{Let } \frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x} \quad \dots(1)$$

$$\Rightarrow 1 = A(1-x^2) + Bx(1+x) + Cx(1-x)$$

$$\Rightarrow 1 = A - Ax^2 + Bx + Bx^2 + Cx - Cx^2$$

Equating the coefficients of  $x^2$ ,  $x$ , and constant term, we obtain

$$-A + B - C = 0$$

$$B + C = 0$$

$$A = 1$$

On solving these equations, we obtain

$$A = 1, B = \frac{1}{2}, \text{ and } C = -\frac{1}{2}$$

From equation (1), we obtain

$$\begin{aligned}\frac{1}{x(1-x)(1+x)} &= \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)} \\ \Rightarrow \int \frac{1}{x(1-x)(1+x)} dx &= \int \frac{1}{x} dx + \frac{1}{2} \int \frac{1}{1-x} dx - \frac{1}{2} \int \frac{1}{1+x} dx \\ &= \log|x| - \frac{1}{2} \log|(1-x)| - \frac{1}{2} \log|(1+x)| \\ &= \log|x| - \log|(1-x)^{\frac{1}{2}}| - \log|(1+x)^{\frac{1}{2}}| \\ &= \log \left| \frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}} \right| + C \\ &= \log \left| \left( \frac{x^2}{1-x^2} \right)^{\frac{1}{2}} \right| + C \\ &= \frac{1}{2} \log \left| \frac{x^2}{1-x^2} \right| + C\end{aligned}$$

**Question 2:**

$$\frac{1}{\sqrt{x+a} + \sqrt{(x+b)}}$$

Answer

$$\begin{aligned} \frac{1}{\sqrt{x+a}+\sqrt{x+b}} &= \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}} \\ &= \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)} \\ &= \frac{(\sqrt{x+a}-\sqrt{x+b})}{a-b} \end{aligned}$$

$$\begin{aligned} \Rightarrow \int \frac{1}{\sqrt{x+a}-\sqrt{x+b}} dx &= \frac{1}{a-b} \int (\sqrt{x+a}-\sqrt{x+b}) dx \\ &= \frac{1}{(a-b)} \left[ \frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}} \right] \\ &= \frac{2}{3(a-b)} \left[ (x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}} \right] + C \end{aligned}$$

**Question 3:**

$$\frac{1}{x\sqrt{ax-x^2}} \quad [\text{Hint: Put } x = \frac{a}{t}]$$

Answer

$$\frac{1}{x\sqrt{ax-x^2}}$$

$$\text{Let } x = \frac{a}{t} \Rightarrow dx = -\frac{a}{t^2} dt$$

$$\begin{aligned} \Rightarrow \int \frac{1}{x\sqrt{ax-x^2}} dx &= \int \frac{1}{\frac{a}{t} \sqrt{a \cdot \frac{a}{t} - \left(\frac{a}{t}\right)^2}} \left(-\frac{a}{t^2} dt\right) \\ &= -\int \frac{1}{at} \cdot \frac{1}{\sqrt{\frac{1}{t} - \frac{1}{t^2}}} dt \\ &= -\frac{1}{a} \int \frac{1}{\sqrt{\frac{t^2}{t} - \frac{1}{t^2}}} dt \\ &= -\frac{1}{a} \int \frac{1}{\sqrt{t-1}} dt \\ &= -\frac{1}{a} [2\sqrt{t-1}] + C \\ &= -\frac{1}{a} \left[2\sqrt{\frac{a}{x}-1}\right] + C \\ &= -\frac{2}{a} \left(\frac{\sqrt{a-x}}{\sqrt{x}}\right) + C \\ &= -\frac{2}{a} \left(\sqrt{\frac{a-x}{x}}\right) + C \end{aligned}$$

**Question 4:**

$$\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$$

Answer

$$\frac{1}{x^2(x^4+1)^{\frac{3}{4}}}$$

Multiplying and dividing by  $x^{-3}$ , we obtain

$$\begin{aligned}\frac{x^{-3}}{x^2 \cdot x^{-3}(x^4+1)^{\frac{3}{4}}} &= \frac{x^{-3}(x^4+1)^{-\frac{3}{4}}}{x^2 \cdot x^{-3}} \\ &= \frac{(x^4+1)^{-\frac{3}{4}}}{x^5 \cdot (x^4)^{-\frac{3}{4}}} \\ &= \frac{1}{x^5} \left( \frac{x^4+1}{x^4} \right)^{-\frac{3}{4}} \\ &= \frac{1}{x^5} \left( 1 + \frac{1}{x^4} \right)^{-\frac{3}{4}}\end{aligned}$$

$$\text{Let } \frac{1}{x^4} = t \Rightarrow -\frac{4}{x^5} dx = dt \Rightarrow \frac{1}{x^5} dx = -\frac{dt}{4}$$

$$\begin{aligned}\therefore \int \frac{1}{x^2(x^4+1)^{\frac{3}{4}}} dx &= \int \frac{1}{x^5} \left( 1 + \frac{1}{x^4} \right)^{-\frac{3}{4}} dx \\ &= -\frac{1}{4} \int (1+t)^{-\frac{3}{4}} dt \\ &= -\frac{1}{4} \left[ \frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}} \right] + C \\ &= -\frac{1}{4} \frac{\left( 1 + \frac{1}{x^4} \right)^{\frac{1}{4}}}{\frac{1}{4}} + C \\ &= -\left( 1 + \frac{1}{x^4} \right)^{\frac{1}{4}} + C\end{aligned}$$

**Question 5:**

$$\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} \left[ \text{Hint: } \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}} \left( 1 + x^{\frac{1}{6}} \right)} \text{ Put } x = t^6 \right]$$

Answer

$$\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}} \left( 1 + x^{\frac{1}{6}} \right)}$$

$$\text{Let } x = t^6 \Rightarrow dx = 6t^5 dt$$

$$\begin{aligned} \therefore \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx &= \int \frac{1}{x^{\frac{1}{3}} \left( 1 + x^{\frac{1}{6}} \right)} dx \\ &= \int \frac{6t^5}{t^2(1+t)} dt \\ &= 6 \int \frac{t^3}{(1+t)} dt \end{aligned}$$

On dividing, we obtain

$$\begin{aligned} \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx &= 6 \int \left\{ (t^2 - t + 1) - \frac{1}{1+t} \right\} dt \\ &= 6 \left[ \left( \frac{t^3}{3} \right) - \left( \frac{t^2}{2} \right) + t - \log|1+t| \right] \\ &= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6 \log \left( 1 + x^{\frac{1}{6}} \right) + C \\ &= 2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6 \log \left( 1 + x^{\frac{1}{6}} \right) + C \end{aligned}$$

**Question 6:**

$$\frac{5x}{(x+1)(x^2+9)}$$

Answer

$$\text{Let } \frac{5x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9} \quad \dots(1)$$

$$\Rightarrow 5x = A(x^2+9) + (Bx+C)(x+1)$$

$$\Rightarrow 5x = Ax^2 + 9A + Bx^2 + Bx + Cx + C$$

Equating the coefficients of  $x^2$ ,  $x$ , and constant term, we obtain

$$A + B = 0$$

$$B + C = 5$$

$$9A + C = 0$$

On solving these equations, we obtain

$$A = -\frac{1}{2}, B = \frac{1}{2}, \text{ and } C = \frac{9}{2}$$

From equation (1), we obtain

$$\frac{5x}{(x+1)(x^2+9)} = \frac{-1}{2(x+1)} + \frac{\frac{x}{2} + \frac{9}{2}}{x^2+9}$$

$$\begin{aligned} \int \frac{5x}{(x+1)(x^2+9)} dx &= \int \left[ \frac{-1}{2(x+1)} + \frac{(x+9)}{2(x^2+9)} \right] dx \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{2} \int \frac{x}{x^2+9} dx + \frac{9}{2} \int \frac{1}{x^2+9} dx \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{4} \int \frac{2x}{x^2+9} dx + \frac{9}{2} \int \frac{1}{x^2+9} dx \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{4} \log|x^2+9| + \frac{9}{2} \cdot \frac{1}{3} \tan^{-1} \frac{x}{3} \\ &= -\frac{1}{2} \log|x+1| + \frac{1}{4} \log(x^2+9) + \frac{3}{2} \tan^{-1} \frac{x}{3} + C \end{aligned}$$

**Question 7:**

$$\frac{\sin x}{\sin(x-a)}$$

Answer

$$\frac{\sin x}{\sin(x-a)}$$

Let  $x - a = t \Rightarrow dx = dt$

$$\begin{aligned} \int \frac{\sin x}{\sin(x-a)} dx &= \int \frac{\sin(t+a)}{\sin t} dt \\ &= \int \frac{\sin t \cos a + \cos t \sin a}{\sin t} dt \\ &= \int (\cos a + \cot t \sin a) dt \\ &= t \cos a + \sin a \log |\sin t| + C_1 \\ &= (x-a) \cos a + \sin a \log |\sin(x-a)| + C_1 \\ &= x \cos a + \sin a \log |\sin(x-a)| - a \cos a + C_1 \\ &= \sin a \log |\sin(x-a)| + x \cos a + C \end{aligned}$$

**Question 8:**

$$\frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}}$$

Answer

$$\begin{aligned} \frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}} &= \frac{e^{4 \log x} (e^{\log x} - 1)}{e^{2 \log x} (e^{\log x} - 1)} \\ &= e^{2 \log x} \\ &= e^{\log x^2} \\ &= x^2 \\ \therefore \int \frac{e^{5 \log x} - e^{4 \log x}}{e^{3 \log x} - e^{2 \log x}} dx &= \int x^2 dx = \frac{x^3}{3} + C \end{aligned}$$

**Question 9:**

$$\frac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Answer



$$\frac{\cos x}{\sqrt{4 - \sin^2 x}}$$

Let  $\sin x = t \Rightarrow \cos x \, dx = dt$

$$\begin{aligned} \Rightarrow \int \frac{\cos x}{\sqrt{4 - \sin^2 x}} dx &= \int \frac{dt}{\sqrt{(2)^2 - (t)^2}} \\ &= \sin^{-1} \left( \frac{t}{2} \right) + C \\ &= \sin^{-1} \left( \frac{\sin x}{2} \right) + C \end{aligned}$$

**Question 10:**

$$\frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x}$$

Answer

$$\begin{aligned} \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x} &= \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{\sin^2 x + \cos^2 x - \sin^2 x \cos^2 x - \sin^2 x \cos^2 x} \\ &= \frac{(\sin^4 x + \cos^4 x)(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{(\sin^2 x - \sin^2 x \cos^2 x) + (\cos^2 x - \sin^2 x \cos^2 x)} \\ &= \frac{(\sin^4 x + \cos^4 x)(\sin^2 x - \cos^2 x)}{\sin^2 x(1 - \cos^2 x) + \cos^2 x(1 - \sin^2 x)} \\ &= \frac{-(\sin^4 x + \cos^4 x)(\cos^2 x - \sin^2 x)}{(\sin^4 x + \cos^4 x)} \\ &= -\cos 2x \end{aligned}$$

$$\therefore \int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x} dx = \int -\cos 2x \, dx = -\frac{\sin 2x}{2} + C$$

**Question 11:**

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Answer

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by  $\sin(a-b)$ , we obtain

$$\begin{aligned} & \frac{1}{\sin(a-b)} \left[ \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)} \right] \\ &= \frac{1}{\sin(a-b)} \left[ \frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)} \right] \\ &= \frac{1}{\sin(a-b)} \left[ \frac{\sin(x+a) \cdot \cos(x+b) - \cos(x+a) \sin(x+b)}{\cos(x+a)\cos(x+b)} \right] \\ &= \frac{1}{\sin(a-b)} \left[ \frac{\sin(x+a)}{\cos(x+a)} - \frac{\sin(x+b)}{\cos(x+b)} \right] \\ &= \frac{1}{\sin(a-b)} [\tan(x+a) - \tan(x+b)] \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\cos(x+a)\cos(x+b)} dx &= \frac{1}{\sin(a-b)} \int [\tan(x+a) - \tan(x+b)] dx \\ &= \frac{1}{\sin(a-b)} [-\log|\cos(x+a)| + \log|\cos(x+b)|] + C \\ &= \frac{1}{\sin(a-b)} \log \left| \frac{\cos(x+b)}{\cos(x+a)} \right| + C \end{aligned}$$

**Question 12:**

$$\frac{x^3}{\sqrt{1-x^8}}$$

Answer

$$\frac{x^3}{\sqrt{1-x^8}}$$

Let  $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\begin{aligned}\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}} dx &= \frac{1}{4} \int \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{1}{4} \sin^{-1} t + C \\ &= \frac{1}{4} \sin^{-1}(x^4) + C\end{aligned}$$

**Question 13:**

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Answer

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Let  $e^x = t \Rightarrow e^x dx = dt$

$$\begin{aligned}\Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)} dx &= \int \frac{dt}{(t+1)(t+2)} \\ &= \int \left[ \frac{1}{(t+1)} - \frac{1}{(t+2)} \right] dt \\ &= \log|t+1| - \log|t+2| + C \\ &= \log \left| \frac{t+1}{t+2} \right| + C \\ &= \log \left| \frac{1+e^x}{2+e^x} \right| + C\end{aligned}$$

**Question 14:**

$$\frac{1}{(x^2+1)(x^2+4)}$$

Answer

$$\therefore \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

$$\Rightarrow 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equating the coefficients of  $x^3$ ,  $x^2$ ,  $x$ , and constant term, we obtain

$$A + C = 0$$

$$B + D = 0$$

$$4A + C = 0$$

$$4B + D = 1$$

On solving these equations, we obtain

$$A = 0, B = \frac{1}{3}, C = 0, \text{ and } D = -\frac{1}{3}$$

From equation (1), we obtain

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)}$$

$$\begin{aligned} \int \frac{1}{(x^2+1)(x^2+4)} dx &= \frac{1}{3} \int \frac{1}{x^2+1} dx - \frac{1}{3} \int \frac{1}{x^2+4} dx \\ &= \frac{1}{3} \tan^{-1} x - \frac{1}{3} \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} + C \\ &= \frac{1}{3} \tan^{-1} x - \frac{1}{6} \tan^{-1} \frac{x}{2} + C \end{aligned}$$

### Question 15:

$$\cos^3 x e^{\log \sin x}$$

Answer

$$\cos^3 x e^{\log \sin x} = \cos^3 x \times \sin x$$

$$\text{Let } \cos x = t \quad \square \quad -\sin x dx = dt$$

$$\begin{aligned}
 \Rightarrow \int \cos^3 x e^{\log \sin x} dx &= \int \cos^3 x \sin x dx \\
 &= - \int t \cdot dt \\
 &= -\frac{t^4}{4} + C \\
 &= -\frac{\cos^4 x}{4} + C
 \end{aligned}$$

**Question 16:**

$$e^{3 \log x} (x^4 + 1)^{-1}$$

Answer

$$e^{3 \log x} (x^4 + 1)^{-1} = e^{\log x^3} (x^4 + 1)^{-1} = \frac{x^3}{(x^4 + 1)}$$

$$\text{Let } x^4 + 1 = t \Rightarrow 4x^3 dx = dt$$

$$\begin{aligned}
 \Rightarrow \int e^{3 \log x} (x^4 + 1)^{-1} dx &= \int \frac{x^3}{(x^4 + 1)} dx \\
 &= \frac{1}{4} \int \frac{dt}{t} \\
 &= \frac{1}{4} \log |t| + C \\
 &= \frac{1}{4} \log |x^4 + 1| + C \\
 &= \frac{1}{4} \log (x^4 + 1) + C
 \end{aligned}$$

**Question 17:**

$$f'(ax+b)[f(ax+b)]^n$$

Answer

$$f'(ax+b)[f(ax+b)]^n$$

$$\text{Let } f(ax+b) = t \Rightarrow af'(ax+b)dx = dt$$

$$\begin{aligned}\Rightarrow \int f'(ax+b)[f(ax+b)]^n dx &= \frac{1}{a} \int t^n dt \\ &= \frac{1}{a} \left[ \frac{t^{n+1}}{n+1} \right] \\ &= \frac{1}{a(n+1)} (f(ax+b))^{n+1} + C\end{aligned}$$

**Question 18:**

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$$

Answer

$$\begin{aligned} \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} &= \frac{1}{\sqrt{\sin^3 x (\sin x \cos \alpha + \cos x \sin \alpha)}} \\ &= \frac{1}{\sqrt{\sin^4 x \cos \alpha + \sin^3 x \cos x \sin \alpha}} \\ &= \frac{1}{\sin^2 x \sqrt{\cos \alpha + \cot x \sin \alpha}} \\ &= \frac{\operatorname{cosec}^2 x}{\sqrt{\cos \alpha + \cot x \sin \alpha}} \end{aligned}$$

$$\text{Let } \cos \alpha + \cot x \sin \alpha = t \Rightarrow -\operatorname{cosec}^2 x \sin \alpha \, dx = dt$$

$$\begin{aligned} \therefore \int \frac{1}{\sin^3 x \sin(x+\alpha)} dx &= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos \alpha + \cot x \sin \alpha}} dx \\ &= \frac{-1}{\sin \alpha} \int \frac{dt}{\sqrt{t}} \\ &= \frac{-1}{\sin \alpha} [2\sqrt{t}] + C \\ &= \frac{-1}{\sin \alpha} [2\sqrt{\cos \alpha + \cot x \sin \alpha}] + C \\ &= \frac{-2}{\sin \alpha} \sqrt{\cos \alpha + \frac{\cos x \sin \alpha}{\sin x}} + C \\ &= \frac{-2}{\sin \alpha} \sqrt{\frac{\sin x \cos \alpha + \cos x \sin \alpha}{\sin x}} + C \\ &= -\frac{2}{\sin \alpha} \sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C \end{aligned}$$

**Question 19:**

$$\frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}}, x \in [0, 1]$$

Answer

$$\text{Let } I = \int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx$$

$$\text{It is known that, } \sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x} = \frac{\pi}{2}$$

$$\begin{aligned} \Rightarrow I &= \int \frac{\left(\frac{\pi}{2} - \cos^{-1} \sqrt{x}\right) - \cos^{-1} \sqrt{x}}{\frac{\pi}{2}} dx \\ &= \frac{2\pi}{\pi} \int \left(\frac{\pi}{2} - 2 \cos^{-1} \sqrt{x}\right) dx \\ &= \frac{2\pi}{\pi} \int 1 \cdot dx - \frac{4}{\pi} \int \cos^{-1} \sqrt{x} dx \\ &= x - \frac{4}{\pi} \int \cos^{-1} \sqrt{x} dx \quad \dots(1) \end{aligned}$$

$$\text{Let } I_1 = \int \cos^{-1} \sqrt{x} dx$$

$$\text{Also, let } \sqrt{x} = t \Rightarrow dx = 2t dt$$

$$\begin{aligned} \Rightarrow I_1 &= 2 \int \cos^{-1} t \cdot t dt \\ &= 2 \left[ \cos^{-1} t \cdot \frac{t^2}{2} - \int \frac{-1}{\sqrt{1-t^2}} \cdot \frac{t^2}{2} dt \right] \\ &= t^2 \cos^{-1} t + \int \frac{t^2}{\sqrt{1-t^2}} dt \\ &= t^2 \cos^{-1} t - \int \frac{1-t^2-1}{\sqrt{1-t^2}} dt \\ &= t^2 \cos^{-1} t - \int \sqrt{1-t^2} dt + \int \frac{1}{\sqrt{1-t^2}} dt \\ &= t^2 \cos^{-1} t - \frac{t}{2} \sqrt{1-t^2} - \frac{1}{2} \sin^{-1} t + \sin^{-1} t \\ &= t^2 \cos^{-1} t - \frac{t}{2} \sqrt{1-t^2} + \frac{1}{2} \sin^{-1} t \end{aligned}$$

From equation (1), we obtain



$$\begin{aligned}
 I &= x - \frac{4}{\pi} \left[ t^2 \cos t - \frac{t}{2} \sqrt{1-t^2} + \frac{1}{2} \sin^{-1} t \right] \\
 &= x - \frac{4}{\pi} \left[ x \cos^{-1} \sqrt{x} - \frac{\sqrt{x}}{2} \sqrt{1-x} + \frac{1}{2} \sin^{-1} \sqrt{x} \right] \\
 &= x - \frac{4\pi}{\pi} \left[ x \left( \frac{-}{2} - \sin^{-1} \sqrt{x} \right) - \frac{\sqrt{x-x^2}}{2} + \frac{1}{2} \sin^{-1} \sqrt{x} \right] \\
 &= x - 2x + \frac{4x}{\pi} \sin^{-1} \sqrt{x} + \frac{2}{\pi} \sqrt{x-x^2} - \frac{2}{\pi} \sin^{-1} \sqrt{x} \\
 &= -x + \frac{2}{\pi} \left[ (2x-1) \sin^{-1} \sqrt{x} \right] + \frac{2}{\pi} \sqrt{x-x^2} + C \\
 &= \frac{2(2x-1)}{\pi} \sin^{-1} \sqrt{x} + \frac{2}{\pi} \sqrt{x-x^2} - x + C
 \end{aligned}$$

**Question 20:**

$$\frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}}$$

Answer

$$I = \int \frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}} dx$$

$$\text{Let } x = \cos^2 \theta \Rightarrow dx = -2 \sin \theta \cos \theta d\theta$$

$$I = \int \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} (-2 \sin \theta \cos \theta) d\theta$$

$$= - \int \sqrt{\frac{2 \sin^2 \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}} \sin 2\theta d\theta$$

$$= - \int \tan \frac{\theta}{2} \cdot 2 \sin \theta \cos \theta d\theta$$

$$= -2 \int \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \left( 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) \cos \theta d\theta$$

$$\begin{aligned}
&= -4 \int \sin^2 \frac{\theta}{2} \cos \theta d\theta \\
&= -4 \int \sin^2 \frac{\theta}{2} \cdot \left( 2 \cos^2 \frac{\theta}{2} - 1 \right) d\theta \\
&= -4 \int \left( 2 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) d\theta \\
&= -8 \int \sin^2 \frac{\theta}{2} \cdot \cos^2 \frac{\theta}{2} d\theta + 4 \int \sin^2 \frac{\theta}{2} d\theta \\
&= -2 \int \sin^2 \theta d\theta + 4 \int \sin^2 \frac{\theta}{2} d\theta \\
&= -2 \int \left( \frac{1 - \cos 2\theta}{2} \right) d\theta + 4 \int \frac{1 - \cos \theta}{2} d\theta \\
&= -2 \left[ \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right] + 4 \left[ \frac{\theta}{2} - \frac{\sin \theta}{2} \right] + C \\
&= -\theta + \frac{\sin 2\theta}{2} + 2\theta - 2 \sin \theta + C \\
&= \theta + \frac{\sin 2\theta}{2} - 2 \sin \theta + C \\
&= \theta + \frac{2 \sin \theta \cos \theta}{2} - 2 \sin \theta + C \\
&= \theta + \sqrt{1 - \cos^2 \theta} \cdot \cos \theta - 2\sqrt{1 - \cos^2 \theta} + C \\
&= \cos^{-1} \sqrt{x} + \sqrt{1-x} \cdot \sqrt{x} - 2\sqrt{1-x} + C \\
&= -2\sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x(1-x)} + C \\
&= -2\sqrt{1-x} + \cos^{-1} \sqrt{x} + \sqrt{x-x^2} + C
\end{aligned}$$

**Question 21:**

$$\frac{2 + \sin 2x}{1 + \cos 2x} e^x$$

Answer

$$\begin{aligned}
 I &= \int \left( \frac{2 + \sin 2x}{1 + \cos 2x} \right) e^x \\
 &= \int \left( \frac{2 + 2 \sin x \cos x}{2 \cos^2 x} \right) e^x \\
 &= \int \left( \frac{1 + \sin x \cos x}{\cos^2 x} \right) e^x \\
 &= \int (\sec^2 x + \tan x) e^x
 \end{aligned}$$

$$\text{Let } f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$$

$$\begin{aligned}
 \therefore I &= \int [f(x) + f'(x)] e^x dx \\
 &= e^x f(x) + C \\
 &= e^x \tan x + C
 \end{aligned}$$

**Question 22:**

$$\frac{x^2 + x + 1}{(x+1)^2(x+2)}$$

Answer

$$\text{Let } \frac{x^2 + x + 1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} \quad \dots(1)$$

$$\Rightarrow x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x^2 + 2x + 1)$$

$$\Rightarrow x^2 + x + 1 = A(x^2 + 3x + 2) + B(x+2) + C(x^2 + 2x + 1)$$

$$\Rightarrow x^2 + x + 1 = (A+C)x^2 + (3A+B+2C)x + (2A+2B+C)$$

Equating the coefficients of  $x^2$ ,  $x$ , and constant term, we obtain

$$A + C = 1$$

$$3A + B + 2C = 1$$

$$2A + 2B + C = 1$$

On solving these equations, we obtain

$$A = -2, B = 1, \text{ and } C = 3$$

From equation (1), we obtain

$$\frac{x^2 + x + 1}{(x+1)^2(x+2)} = \frac{-2}{(x+1)} + \frac{3}{(x+2)} + \frac{1}{(x+1)^2}$$

$$\begin{aligned} \int \frac{x^2 + x + 1}{(x+1)^2(x+2)} dx &= -2 \int \frac{1}{x+1} dx + 3 \int \frac{1}{(x+2)} dx + \int \frac{1}{(x+1)^2} dx \\ &= -2 \log|x+1| + 3 \log|x+2| - \frac{1}{(x+1)} + C \end{aligned}$$

**Question 23:**

$$\tan^{-1} \sqrt{\frac{1-x}{1+x}}$$

Answer

$$I = \int \tan^{-1} \sqrt{\frac{1-x}{1+x}} dx$$

$$\text{Let } x = \cos \theta \Rightarrow dx = -\sin \theta d\theta$$

$$I = \int \tan^{-1} \sqrt{\frac{1-\cos \theta}{1+\cos \theta}} (-\sin \theta d\theta)$$

$$= -\int \tan^{-1} \sqrt{\frac{2 \sin^2 \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}}} \sin \theta d\theta$$

$$= -\int \tan^{-1} \tan \frac{\theta}{2} \cdot \sin \theta d\theta$$

$$= -\frac{1}{2} \int \theta \cdot \sin \theta d\theta$$

$$= -\frac{1}{2} [\theta \cdot (-\cos \theta) - \int 1 \cdot (-\cos \theta) d\theta]$$

$$= -\frac{1}{2} [-\theta \cos \theta + \sin \theta]$$

$$= +\frac{1}{2} \theta \cos \theta - \frac{1}{2} \sin \theta$$

$$= \frac{1}{2} \cos^{-1} x \cdot x - \frac{1}{2} \sqrt{1-x^2} + C$$

$$= \frac{x}{2} \cos^{-1} x - \frac{1}{2} \sqrt{1-x^2} + C$$

$$= \frac{1}{2} (x \cos^{-1} x - \sqrt{1-x^2}) + C$$

**Question 24:**

$$\frac{\sqrt{x^2+1} [\log(x^2+1) - 2 \log x]}{x^4}$$

**Answer**

$$\begin{aligned} \frac{\sqrt{x^2+1} [\log(x^2+1) - 2 \log x]}{x^4} &= \frac{\sqrt{x^2+1}}{x^4} [\log(x^2+1) - \log x^2] \\ &= \frac{\sqrt{x^2+1}}{x^4} \left[ \log \left( \frac{x^2+1}{x^2} \right) \right] \\ &= \frac{\sqrt{x^2+1}}{x^4} \log \left( 1 + \frac{1}{x^2} \right) \\ &= \frac{1}{x^3} \sqrt{\frac{x^2+1}{x^2}} \log \left( 1 + \frac{1}{x^2} \right) \\ &= \frac{1}{x^3} \sqrt{1 + \frac{1}{x^2}} \log \left( 1 + \frac{1}{x^2} \right) \end{aligned}$$

$$\text{Let } 1 + \frac{1}{x^2} = t \Rightarrow \frac{-2}{x^3} dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{1}{x^3} \sqrt{1 + \frac{1}{x^2}} \log \left( 1 + \frac{1}{x^2} \right) dx \\ &= -\frac{1}{2} \int \sqrt{t} \log t \, dt \\ &= -\frac{1}{2} \int t^{\frac{1}{2}} \cdot \log t \, dt \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
 I &= -\frac{1}{2} \left[ \log t \cdot \int t^{\frac{1}{2}} dt - \left\{ \left( \frac{d}{dt} \log t \right) \int t^{\frac{1}{2}} dt \right\} dt \right] \\
 &= -\frac{1}{2} \left[ \log t \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - \int \frac{1}{t} \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} dt \right] \\
 &= -\frac{1}{2} \left[ \frac{2}{3} t^{\frac{3}{2}} \log t - \frac{2}{3} \int t^{\frac{1}{2}} dt \right] \\
 &= -\frac{1}{2} \left[ \frac{2}{3} t^{\frac{3}{2}} \log t - \frac{4}{9} t^{\frac{3}{2}} \right] \\
 &= -\frac{1}{3} t^{\frac{3}{2}} \log t + \frac{2}{9} t^{\frac{3}{2}} \\
 &= -\frac{1}{3} t^{\frac{3}{2}} \left[ \log t - \frac{2}{3} \right] \\
 &= -\frac{1}{3} \left( 1 + \frac{1}{x^2} \right)^{\frac{3}{2}} \left[ \log \left( 1 + \frac{1}{x^2} \right) - \frac{2}{3} \right] + C
 \end{aligned}$$

**Question 25:**

$$\int_{\frac{\pi}{2}}^{\pi} e^x \left( \frac{1 - \sin x}{1 - \cos x} \right) dx$$

Answer

$$\begin{aligned}
 I &= \int_{\frac{\pi}{2}}^{\pi} e^x \left( \frac{1 - \sin x}{1 - \cos x} \right) dx \\
 &= \int_{\frac{\pi}{2}}^{\pi} e^x \left( \frac{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \right) dx \\
 &= \int_{\frac{\pi}{2}}^{\pi} e^x \left( \frac{\operatorname{cosec}^2 \frac{x}{2}}{2} - \cot \frac{x}{2} \right) dx
 \end{aligned}$$

$$\text{Let } f(x) = -\cot \frac{x}{2}$$

$$\Rightarrow f'(x) = -\left( -\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} \right) = \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x (f(x) + f'(x)) dx$$

$$\begin{aligned}
 &= \left[ e^x \cdot f(x) \right]_{\frac{\pi}{2}}^{\pi} \\
 &= -\left[ e^x \cdot \cot \frac{x}{2} \right]_{\frac{\pi}{2}}^{\pi} \\
 &= -\left[ e^{\pi} \times \cot \frac{\pi}{2} - e^{\frac{\pi}{2}} \times \cot \frac{\pi}{4} \right] \\
 &= -\left[ e^{\pi} \times 0 - e^{\frac{\pi}{2}} \times 1 \right] \\
 &= e^{\frac{\pi}{2}}
 \end{aligned}$$

**Question 26:**

$$\int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{(\sin x \cos x)}{\frac{\cos^4 x}{\cos^4 x} + \frac{\sin^4 x}{\cos^4 x}} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \frac{\tan x \sec^2 x}{1 + \tan^4 x} dx$$

$$\text{Let } \tan^2 x = t \Rightarrow 2 \tan x \sec^2 x dx = dt$$

$$\text{When } x = 0, t = 0 \text{ and } \text{when } x = \frac{\pi}{4}, t = 1$$

$$\therefore I = \frac{1}{2} \int_0^1 \frac{dt}{1+t^2}$$

$$= \frac{1}{2} [\tan^{-1} t]_0^1$$

$$= \frac{1}{2} [\tan^{-1} 1 - \tan^{-1} 0]$$

$$= \frac{1}{2} \left[ \frac{\pi}{4} \right]$$

$$= \frac{\pi}{8}$$

**Question 27:**

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x dx}{\cos^2 x + 4 \sin^2 x}$$

Answer



$$\begin{aligned}
 \text{Let } I &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x} dx \\
 \Rightarrow I &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4(1 - \cos^2 x)} dx \\
 \Rightarrow I &= \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos^2 x + 4 - 4 \cos^2 x} dx \\
 \Rightarrow I &= \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3 \cos^2 x - 4}{4 - 3 \cos^2 x} dx \\
 \Rightarrow I &= \frac{-1}{3} \int_0^{\frac{\pi}{2}} \frac{4 - 3 \cos^2 x}{4 - 3 \cos^2 x} dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4}{4 - 3 \cos^2 x} dx \\
 \Rightarrow I &= \frac{-1}{3} \int_0^{\frac{\pi}{2}} 1 dx + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{4 \sec^2 x - 3} dx \\
 \Rightarrow I &= \frac{-1}{3} [x]_0^{\frac{\pi}{2}} + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{4 \sec^2 x}{4(1 + \tan^2 x) - 3} dx \\
 \Rightarrow I &= -\frac{\pi}{6} + \frac{2}{3} \int_0^{\frac{\pi}{2}} \frac{2 \sec^2 x}{1 + 4 \tan^2 x} dx \quad \dots(1)
 \end{aligned}$$

Consider,  $\int_0^{\frac{\pi}{2}} \frac{2 \sec^2 x}{1 + 4 \tan^2 x} dx$

Let  $2 \tan x = t \Rightarrow 2 \sec^2 x dx = dt$

When  $x = 0, t = 0$  and when  $x = \frac{\pi}{2}, t = \infty$

$$\begin{aligned}
 \Rightarrow \int_0^{\frac{\pi}{2}} \frac{2 \sec^2 x}{1 + 4 \tan^2 x} dx &= \int_0^{\infty} \frac{dt}{1 + t^2} \\
 &= [\tan^{-1} t]_0^{\infty} \\
 &= [\tan^{-1}(\infty) - \tan^{-1}(0)] \\
 &= \frac{\pi}{2}
 \end{aligned}$$

Therefore, from (1), we obtain

$$I = -\frac{\pi}{6} + \frac{2}{3} \left[ \frac{\pi}{2} \right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

**Question 28:**

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Answer

$$\text{Let } I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{-(-\sin 2x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2\sin x \cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin^2 x + \cos^2 x - 2\sin x \cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x) dx}{\sqrt{1-(\sin x - \cos x)^2}}$$

$$\text{Let } (\sin x - \cos x) = t \Rightarrow (\sin x + \cos x) dx = dt$$

$$\text{When } x = \frac{\pi}{6}, t = \left(\frac{1-\sqrt{3}}{2}\right) \text{ and when } x = \frac{\pi}{3}, t = \left(\frac{\sqrt{3}-1}{2}\right)$$

$$I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I = \int_{\left(\frac{\sqrt{3}-1}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\text{As } \frac{1}{\sqrt{1-(-t)^2}} = \frac{1}{\sqrt{1-t^2}}, \text{ therefore, } \frac{1}{\sqrt{1-t^2}} \text{ is an even function.}$$

$$\text{It is known that if } f(x) \text{ is an even function, then } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

$$\begin{aligned}
 \Rightarrow I &= 2 \int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}} \\
 &= \left[ 2 \sin^{-1} t \right]_0^{\frac{\sqrt{3}-1}{2}} \\
 &= 2 \sin^{-1} \left( \frac{\sqrt{3}-1}{2} \right)
 \end{aligned}$$

**Question 29:**

$$\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Answer

$$\begin{aligned}
 \text{Let } I &= \int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}} \\
 I &= \int_0^1 \frac{1}{(\sqrt{1+x}-\sqrt{x})} \times \frac{(\sqrt{1+x}+\sqrt{x})}{(\sqrt{1+x}+\sqrt{x})} dx \\
 &= \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx \\
 &= \int_0^1 \sqrt{1+x} dx + \int_0^1 \sqrt{x} dx \\
 &= \left[ \frac{2}{3}(1+x)^{\frac{3}{2}} \right]_0^1 + \left[ \frac{2}{3}(x)^{\frac{3}{2}} \right]_0^1 \\
 &= \frac{2}{3} \left[ (2)^{\frac{3}{2}} - 1 \right] + \frac{2}{3} [1] \\
 &= \frac{2}{3} (2)^{\frac{3}{2}} \\
 &= \frac{2 \cdot 2\sqrt{2}}{3} \\
 &= \frac{4\sqrt{2}}{3}
 \end{aligned}$$

**Question 30:**

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

$$\text{Also, let } \sin x - \cos x = t \Rightarrow (\cos x + \sin x) dx = dt$$

$$\text{When } x = 0, t = -1 \text{ and when } x = \frac{\pi}{4}, t = 0$$

$$\Rightarrow (\sin x - \cos x)^2 = t^2$$

$$\Rightarrow \sin^2 x + \cos^2 x - 2 \sin x \cos x = t^2$$

$$\Rightarrow 1 - \sin 2x = t^2$$

$$\Rightarrow \sin 2x = 1 - t^2$$

$$\begin{aligned} \therefore I &= \int_{-1}^0 \frac{dt}{9 + 16(1 - t^2)} \\ &= \int_{-1}^0 \frac{dt}{9 + 16 - 16t^2} \\ &= \int_{-1}^0 \frac{dt}{25 - 16t^2} = \int_{-1}^0 \frac{dt}{(5)^2 - (4t)^2} \\ &= \frac{1}{4} \left[ \frac{1}{2(5)} \log \left| \frac{5+4t}{5-4t} \right| \right]_{-1}^0 \\ &= \frac{1}{40} \left[ \log(1) - \log \left| \frac{1}{9} \right| \right] \\ &= \frac{1}{40} \log 9 \end{aligned}$$

**Question 31:**

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx = \int_0^{\frac{\pi}{2}} 2 \sin x \cos x \tan^{-1}(\sin x) dx$$

$$\text{Also, let } \sin x = t \Rightarrow \cos x dx = dt$$

$$\text{When } x = 0, t = 0 \text{ and when } x = \frac{\pi}{2}, t = 1$$

$$\Rightarrow I = 2 \int_0^1 t \tan^{-1}(t) dt \quad \dots(1)$$

$$\text{Consider } \int t \cdot \tan^{-1} t dt = \tan^{-1} t \cdot \int t dt - \int \left\{ \frac{d}{dt}(\tan^{-1} t) \int t dt \right\} dt$$

$$\begin{aligned} &= \tan^{-1} t \cdot \frac{t^2}{2} - \int \frac{1}{1+t^2} \cdot \frac{t^2}{2} dt \\ &= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2} \int \frac{t^2 + 1 - 1}{1+t^2} dt \\ &= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2} \int 1 dt + \frac{1}{2} \int \frac{1}{1+t^2} dt \\ &= \frac{t^2 \tan^{-1} t}{2} - \frac{1}{2} \cdot t + \frac{1}{2} \tan^{-1} t \end{aligned}$$

$$\begin{aligned} \Rightarrow \int_0^1 t \cdot \tan^{-1} t dt &= \left[ \frac{t^2 \cdot \tan^{-1} t}{2} - \frac{t}{2} + \frac{1}{2} \tan^{-1} t \right]_0^1 \\ &= \frac{1}{2} \left[ \frac{\pi}{4} - 1 + \frac{\pi}{4} \right] \\ &= \frac{1}{2} \left[ \frac{\pi}{2} - 1 \right] = \frac{\pi}{4} - \frac{1}{2} \end{aligned}$$

From equation (1), we obtain

$$I = 2 \left[ \frac{\pi}{4} - \frac{1}{2} \right] = \frac{\pi}{2} - 1$$

**Question 32:**

$$\int_0^{\frac{\pi}{2}} \frac{x \tan x}{\sec x + \tan x} dx$$

Answer

$$\text{Let } I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx \quad \dots(1)$$

$$I = \int_0^{\pi} \left\{ \frac{(\pi - x) \tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} \right\} dx \quad \left( \int_0^a f(x) dx = \int_0^a f(a - x) dx \right)$$

$$\Rightarrow I = \int_0^{\pi} \left\{ \frac{-(\pi - x) \tan x}{-(\sec x + \tan x)} \right\} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x) \tan x}{\sec x + \tan x} dx \quad \dots(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\frac{\sin x}{\cos x}}{1 + \frac{\sin x}{\cos x}} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x + 1 - 1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} 1 \cdot dx - \pi \int_0^{\pi} \frac{1}{1 + \sin x} dx$$

$$\Rightarrow 2I = \pi [x]_0^{\pi} - \pi \int_0^{\pi} \frac{1 - \sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi^2 - \pi \int_0^{\pi} (\sec^2 x - \tan x \sec x) dx$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan x - \sec x]_0^{\pi}$$

$$\Rightarrow 2I = \pi^2 - \pi [\tan \pi - \sec \pi - \tan 0 + \sec 0]$$

$$\Rightarrow 2I = \pi^2 - \pi [0 - (-1) - 0 + 1]$$

$$\Rightarrow 2I = \pi^2 - 2\pi$$

$$\Rightarrow 2I = \pi(\pi - 2)$$

$$\Rightarrow I = \frac{\pi}{2}(\pi - 2)$$

**Question 33:**

$$\int_1^4 [|x-1| + |x-2| + |x-3|] dx$$

Answer

$$\text{Let } I = \int_1^4 [|x-1| + |x-2| + |x-3|] dx$$

$$\Rightarrow I = \int_1^4 |x-1| dx + \int_1^4 |x-2| dx + \int_1^4 |x-3| dx$$

$$I = I_1 + I_2 + I_3 \quad \dots(1)$$

$$\text{where, } I_1 = \int_1^4 |x-1| dx, I_2 = \int_1^4 |x-2| dx, \text{ and } I_3 = \int_1^4 |x-3| dx$$

$$I_1 = \int_1^4 |x-1| dx$$

$$(x-1) \geq 0 \text{ for } 1 \leq x \leq 4$$

$$\therefore I_1 = \int_1^4 (x-1) dx$$

$$\Rightarrow I_1 = \left[ \frac{x^2}{2} - x \right]_1^4$$

$$\Rightarrow I_1 = \left[ 8 - 4 - \frac{1}{2} + 1 \right] = \frac{9}{2} \quad \dots(2)$$

$$I_2 = \int_1^4 |x-2| dx$$

$$x-2 \geq 0 \text{ for } 2 \leq x \leq 4 \text{ and } x-2 \leq 0 \text{ for } 1 \leq x \leq 2$$

$$\therefore I_2 = \int_1^2 (2-x) dx + \int_2^4 (x-2) dx$$

$$\Rightarrow I_2 = \left[ 2x - \frac{x^2}{2} \right]_1^2 + \left[ \frac{x^2}{2} - 2x \right]_2^4$$

$$\Rightarrow I_2 = \left[ 4 - 2 - 2 + \frac{1}{2} \right] + [8 - 8 - 2 + 4]$$

$$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2} \quad \dots(3)$$

$$I_3 = \int_1^4 |x-3| dx$$

$x-3 \geq 0$  for  $3 \leq x \leq 4$  and  $x-3 \leq 0$  for  $1 \leq x \leq 3$

$$\therefore I_3 = \int_1^3 (3-x) dx + \int_3^4 (x-3) dx$$

$$\Rightarrow I_3 = \left[ 3x - \frac{x^2}{2} \right]_1^3 + \left[ \frac{x^2}{2} - 3x \right]_3^4$$

$$\Rightarrow I_3 = \left[ 9 - \frac{9}{2} - 3 + \frac{1}{2} \right] + \left[ 8 - 12 - \frac{9}{2} + 9 \right]$$

$$\Rightarrow I_3 = [6-4] + \left[ \frac{1}{2} \right] = \frac{5}{2} \quad \dots(4)$$

From equations (1), (2), (3), and (4), we obtain

$$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

**Question 34:**

$$\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log \frac{2}{3}$$

Answer

$$\text{Let } I = \int_1^3 \frac{dx}{x^2(x+1)}$$

$$\text{Also, let } \frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

Equating the coefficients of  $x^2$ ,  $x$ , and constant term, we obtain

$$A + C = 0$$

$$A + B = 0$$

$$B = 1$$

On solving these equations, we obtain

$$A = -1, C = 1, \text{ and } B = 1$$



$$\begin{aligned}
 \therefore \frac{1}{x^2(x+1)} &= \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \\
 \Rightarrow I &= \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx \\
 &= \left[ -\log x - \frac{1}{x} + \log(x+1) \right]_1^3 \\
 &= \left[ \log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3 \\
 &= \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1 \\
 &= \log 4 - \log 3 - \log 2 + \frac{2}{3} \\
 &= \log 2 - \log 3 + \frac{2}{3} \\
 &= \log\left(\frac{2}{3}\right) + \frac{2}{3}
 \end{aligned}$$

Hence, the given result is proved.

**Question 35:**

$$\int_0^1 xe^x dx = 1$$

Answer

$$\text{Let } I = \int_0^1 xe^x dx$$

Integrating by parts, we obtain

$$\begin{aligned}
 I &= x \int_0^1 e^x dx - \int_0^1 \left\{ \left( \frac{d}{dx}(x) \right) \int e^x dx \right\} dx \\
 &= [xe^x]_0^1 - \int_0^1 e^x dx \\
 &= [xe^x]_0^1 - [e^x]_0^1 \\
 &= e - e + 1 \\
 &= 1
 \end{aligned}$$

Hence, the given result is proved.

**Question 36:**

$$\int_{-1}^1 x^{17} \cos^4 x dx = 0$$

Answer

$$\text{Let } I = \int_{-1}^1 x^{17} \cos^4 x dx$$

$$\text{Also, let } f(x) = x^{17} \cos^4 x$$

$$\Rightarrow f(-x) = (-x)^{17} \cos^4(-x) = -x^{17} \cos^4 x = -f(x)$$

Therefore,  $f(x)$  is an odd function.

It is known that if  $f(x)$  is an odd function, then  $\int_{-a}^a f(x) dx = 0$

$$\therefore I = \int_{-1}^1 x^{17} \cos^4 x dx = 0$$

Hence, the given result is proved.

**Question 37:**

$$\int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sin^3 x dx$$

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \sin x dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \sin x dx$$

$$= \int_0^{\frac{\pi}{2}} \sin x dx - \int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin x dx$$

$$= [-\cos x]_0^{\frac{\pi}{2}} + \left[ \frac{\cos^3 x}{3} \right]_0^{\frac{\pi}{2}}$$

$$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$$

Hence, the given result is proved.

**Question 38:**

$$\int_0^{\frac{\pi}{4}} 2 \tan^3 x dx = 1 - \log 2$$

Answer

$$\text{Let } I = \int_0^{\frac{\pi}{4}} 2 \tan^3 x dx$$

$$I = 2 \int_0^{\frac{\pi}{4}} \tan^2 x \tan x dx = 2 \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \tan x dx$$

$$= 2 \int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx - 2 \int_0^{\frac{\pi}{4}} \tan x dx$$

$$= 2 \left[ \frac{\tan^2 x}{2} \right]_0^{\frac{\pi}{4}} + 2 \left[ \log \cos x \right]_0^{\frac{\pi}{4}}$$

$$= 1 + 2 \left[ \log \cos \frac{\pi}{4} - \log \cos 0 \right]$$

$$= 1 + 2 \left[ \log \frac{1}{\sqrt{2}} - \log 1 \right]$$

$$= 1 - \log 2 - \log 1 = 1 - \log 2$$

Hence, the given result is proved.

**Question 39:**

$$\int_0^1 \sin^{-1} x dx = \frac{\pi}{2} - 1$$

Answer

$$\text{Let } I = \int_0^1 \sin^{-1} x dx$$

$$\Rightarrow I = \int_0^1 \sin^{-1} x \cdot 1 \cdot dx$$

Integrating by parts, we obtain

$$\begin{aligned}
 I &= \left[ \sin^{-1} x \cdot x \right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot x \, dx \\
 &= \left[ x \sin^{-1} x \right]_0^1 + \frac{1}{2} \int_0^1 \frac{(-2x)}{\sqrt{1-x^2}} \, dx
 \end{aligned}$$

Let  $1 - x^2 = t \Rightarrow -2x \, dx = dt$

When  $x = 0$ ,  $t = 1$  and when  $x = 1$ ,  $t = 0$

$$\begin{aligned}
 I &= \left[ x \sin^{-1} x \right]_0^1 + \frac{1}{2} \int_1^0 \frac{dt}{\sqrt{t}} \\
 &= \left[ x \sin^{-1} x \right]_0^1 + \frac{1}{2} \left[ 2\sqrt{t} \right]_1^0 \\
 &= \sin^{-1}(1) + \left[ -\sqrt{1} \right] \\
 &= \frac{\pi}{2} - 1
 \end{aligned}$$

Hence, the given result is proved.

#### Question 40:

Evaluate  $\int_0^1 e^{2-3x} \, dx$  as a limit of a sum.

Answer

$$\text{Let } I = \int_0^1 e^{2-3x} \, dx$$

It is known that,

$$\int_a^b f(x) \, dx = (b-a) \lim_{n \rightarrow \infty} \frac{1}{n} \left[ f(a) + f(a+h) + \dots + f(a+(n-1)h) \right]$$

$$\text{Where, } h = \frac{b-a}{n}$$

Here,  $a = 0$ ,  $b = 1$ , and  $f(x) = e^{2-3x}$

$$\Rightarrow h = \frac{1-0}{n} = \frac{1}{n}$$

$$\begin{aligned}
 \therefore \int_0^1 e^{2-3x} \, dx &= (1-0) \lim_{n \rightarrow \infty} \frac{1}{n} \left[ f(0) + f(0+h) + \dots + f(0+(n-1)h) \right] \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[ e^2 + e^{2-3h} + \dots + e^{2-3(n-1)h} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \frac{1}{n} \left[ e^2 \left\{ 1 + e^{-3h} + e^{-6h} + e^{-9h} + \dots + e^{-3(n-1)h} \right\} \right] \\
&= \lim_{h \rightarrow \infty} \frac{1}{n} \left[ e^2 \left\{ \frac{1 - (e^{-3h})^n}{1 - e^{-3h}} \right\} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \left[ e^2 \left\{ \frac{1 - e^{-\frac{3}{n} \times n}}{1 - e^{-\frac{3}{n}}} \right\} \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \frac{e^2 (1 - e^{-3})}{1 - e^{-\frac{3}{n}}} \right] \\
&= e^2 (e^{-3} - 1) \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \frac{1}{e^{-\frac{3}{n}} - 1} \right] \\
&= e^2 (e^{-3} - 1) \lim_{n \rightarrow \infty} \left( -\frac{1}{3} \right) \left[ \frac{-\frac{3}{n}}{e^{-\frac{3}{n}} - 1} \right] \\
&= \frac{-e^2 (e^{-3} - 1)}{3} \lim_{n \rightarrow \infty} \left[ \frac{-\frac{3}{n}}{e^{-\frac{3}{n}} - 1} \right] \\
&= \frac{-e^2 (e^{-3} - 1)}{3} (1) \quad \left[ \lim_{n \rightarrow \infty} \frac{x}{e^x - 1} \right] \\
&= \frac{-e^{-1} + e^2}{3} \\
&= \frac{1}{3} \left( e^2 - \frac{1}{e} \right)
\end{aligned}$$

**Question 41:**

$\int \frac{dx}{e^x + e^{-x}}$  is equal to

**A.**  $\tan^{-1}(e^x) + C$

- B.**  $\tan^{-1}(e^{-x}) + C$   
**C.**  $\log(e^x - e^{-x}) + C$   
**D.**  $\log(e^x + e^{-x}) + C$

Answer

$$\text{Let } I = \int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x}{e^{2x} + 1} dx$$

$$\text{Also, let } e^x = t \Rightarrow e^x dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{dt}{1+t^2} \\ &= \tan^{-1} t + C \\ &= \tan^{-1}(e^x) + C \end{aligned}$$

Hence, the correct Answer is A.

**Question 42:**

$$\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx \text{ is equal to}$$

- A.**  $\frac{-1}{\sin x + \cos x} + C$   
**B.**  $\log|\sin x + \cos x| + C$   
**C.**  $\log|\sin x - \cos x| + C$   
**D.**  $\frac{1}{(\sin x + \cos x)^2}$

Answer

$$\begin{aligned} \text{Let } I &= \frac{\cos 2x}{(\cos x + \sin x)^2} \\ I &= \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)^2} dx \\ &= \int \frac{(\cos x + \sin x)(\cos x - \sin x)}{(\cos x + \sin x)^2} dx \\ &= \int \frac{\cos x - \sin x}{\cos x + \sin x} dx \end{aligned}$$

$$\text{Let } \cos x + \sin x = t \Rightarrow (\cos x - \sin x) dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{dt}{t} \\ &= \log|t| + C \\ &= \log|\cos x + \sin x| + C \end{aligned}$$

Hence, the correct Answer is B.

**Question 43:**

If  $f(a+b-x) = f(x)$ , then  $\int_a^b x f(x) dx$  is equal to

- A.**  $\frac{a+b}{2} \int_a^b f(b-x) dx$
- B.**  $\frac{a+b}{2} \int_a^b f(b+x) dx$
- C.**  $\frac{b-a}{2} \int_a^b f(x) dx$
- D.**  $\frac{a+b}{2} \int_a^b f(x) dx$

Answer

$$\text{Let } I = \int_a^b x f(x) dx \quad \dots(1)$$

$$\begin{aligned}
 I &= \int_a^b (a+b-x)f(a+b-x)dx && \left( \int_a^b f(x)dx = \int_a^b f(a+b-x)dx \right) \\
 \Rightarrow I &= \int_a^b (a+b-x)f(x)dx \\
 \Rightarrow I &= (a+b) \int_a^b f(x)dx && -I \quad \quad \quad [\text{Using (1)}] \\
 \Rightarrow I+I &= (a+b) \int_a^b f(x)dx \\
 \Rightarrow 2I &= (a+b) \int_a^b f(x)dx \\
 \Rightarrow I &= \left( \frac{a+b}{2} \right) \int_a^b f(x)dx
 \end{aligned}$$

Hence, the correct Answer is D.

**Question 44:**

The value of  $\int_0^1 \tan^{-1} \left( \frac{2x-1}{1+x-x^2} \right) dx$  is

- A. 1
- B. 0
- C. - 1
- D.  $\frac{\pi}{4}$

Answer

$$\begin{aligned}
 \text{Let } I &= \int_0^1 \tan^{-1} \left( \frac{2x-1}{1+x-x^2} \right) dx \\
 \Rightarrow I &= \int_0^1 \tan^{-1} \left( \frac{x-(1-x)}{1+x(1-x)} \right) dx \\
 \Rightarrow I &= \int_0^1 [\tan^{-1} x - \tan^{-1} (1-x)] dx && \dots(1) \\
 \Rightarrow I &= \int_0^1 [\tan^{-1} (1-x) - \tan^{-1} (1-1+x)] dx \\
 \Rightarrow I &= \int_0^1 [\tan^{-1} (1-x) - \tan^{-1} (x)] dx \\
 \Rightarrow I &= \int_0^1 [\tan^{-1} (1-x) - \tan^{-1} (x)] dx && \dots(2)
 \end{aligned}$$



Adding (1) and (2), we obtain

$$2I = \int_0^1 (\tan^{-1} x + \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1} x) dx$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

Hence, the correct Answer is B.