Exercise 8.1

Question 1:

Expand the expression $(1-2x)^5$

Answer

By using Binomial Theorem, the expression $(1-2x)^5$ can be expanded as

$$(1-2x)^5$$
$$= {}^5C_0(1)^5 - {}^5C_1(1)^4(2x) + {}^5C_2(1)^3(2x)^2 - {}^5C_3(1)^2(2x)^3 + {}^5C_4(1)^1(2x)^4 - {}^5C_5(2x)^5$$
$$= 1 - 5(2x) + 10(4x^2) - 10(8x^3) + 5(16x^4) - (32x^5)$$
$$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$$

Question 2:

Expand the expression $\left(\frac{2}{x} - \frac{x}{2}\right)^5$

Answer

By using Binomial Theorem, the expression $\left(\frac{2}{x} - \frac{x}{2}\right)^5$ can be expanded as

$$\left(\frac{2}{x} - \frac{x}{2}\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2$$
$$- {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5$$
$$= \frac{32}{x^5} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) - 10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32}$$
$$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}$$

Question 3:

Expand the expression $(2x-3)^6$

Answer

By using Binomial Theorem, the expression $(2x-3)^6$ can be expanded as

$$\begin{aligned}(2x-3)^6 &= {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2 - {}^6C_3(2x)^3(3)^3 \\ &\quad + {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5 + {}^6C_6(3)^6 \\ &= 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27) \\ &\quad + 15(4x^2)(81) - 6(2x)(243) + 729 \\ &= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729\end{aligned}$$

Question 4:

Expand the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

Answer

By using Binomial Theorem, the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ can be expanded as

$$\begin{aligned}\left(\frac{x}{3}+\frac{1}{x}\right)^5 &= {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 \\ &\quad + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5 \\ &= \frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5} \\ &= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}\end{aligned}$$

Question 5:

Expand $\left(x+\frac{1}{x}\right)^6$

Answer

By using Binomial Theorem, the expression $\left(x+\frac{1}{x}\right)^6$ can be expanded as

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0(x)^6 + {}^6C_1(x)^5\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2$$
$$+ {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$
$$= x^6 + 6(x)^5\left(\frac{1}{x}\right) + 15(x)^4\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right) + 15(x)^2\left(\frac{1}{x^4}\right) + 6(x)\left(\frac{1}{x^5}\right) + \frac{1}{x^6}$$
$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Question 6:

Using Binomial Theorem, evaluate $(96)^3$

Answer

96 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, 96 = 100 – 4

$$\therefore (96)^3 = (100-4)^3$$
$$= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)(4)^2 - {}^3C_3(4)^3$$
$$= (100)^3 - 3(100)^2(4) + 3(100)(4)^2 - (4)^3$$
$$= 1000000 - 120000 + 4800 - 64$$
$$= 884736$$

Question 7:

Using Binomial Theorem, evaluate $(102)^5$

Answer

102 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 102 = 100 + 2

$$\therefore (102)^5 = (100+2)^5$$
$$= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3$$
$$+ {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$$
$$= (100)^5 + 5(100)^4(2) + 10(100)^3(2)^2 + 10(100)^2(2)^3 + 5(100)(2)^4 + (2)^5$$
$$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$$
$$= 11040808032$$

Question 8:

Using Binomial Theorem, evaluate $(101)^4$

Answer

101 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 101 = 100 + 1

$$\therefore (101)^4 = (100+1)^4$$
$$= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4$$
$$= (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4$$
$$= 100000000 + 4000000 + 60000 + 400 + 1$$
$$= 104060401$$

Question 9:

Using Binomial Theorem, evaluate $(99)^5$

Answer

99 can be written as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, 99 = 100 – 1

$$\therefore (99)^5 = (100-1)^5$$
$$= {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3$$
$$+ {}^5C_4(100)(1)^4 - {}^5C_5(1)^5$$
$$= (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 5(100) - 1$$
$$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$$
$$= 10010000500 - 500100001$$
$$= 9509900499$$

Question 10:

Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Answer

By splitting 1.1 and then applying Binomial Theorem, the first few terms of $(1.1)^{10000}$ can be obtained as

$$(1.1)^{10000} = (1+0.1)^{10000}$$
$$= {}^{10000}C_0 + {}^{10000}C_1(1.1) + \text{Other positive terms}$$
$$= 1 + 10000 \times 1.1 + \text{Other positive terms}$$
$$= 1 + 11000 + \text{Other positive terms}$$
$$> 1000$$

Hence, $(1.1)^{10000} > 1000$

Question 11:

Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$.

Answer

Using Binomial Theorem, the expressions, $(a + b)^4$ and $(a - b)^4$, can be expanded as

$$(a+b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$$

$$(a-b)^4 = {}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4$$

$$\therefore (a+b)^4 - (a-b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$$

$$-\left[{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4\right]$$

$$= 2\left({}^4C_1a^3b + {}^4C_3ab^3\right) = 2\left(4a^3b + 4ab^3\right)$$

$$= 8ab\left(a^2 + b^2\right)$$

By putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4 = 8\left(\sqrt{3}\right)\left(\sqrt{2}\right)\left\{\left(\sqrt{3}\right)^2 + \left(\sqrt{2}\right)^2\right\}$$

$$= 8\left(\sqrt{6}\right)\left\{3+2\right\} = 40\sqrt{6}$$

Question 12:

Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$.

Answer

Using Binomial Theorem, the expressions, $(x + 1)^6$ and $(x - 1)^6$, can be expanded as

$$(x+1)^6 = {}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_4x^2 + {}^6C_5x + {}^6C_6$$

$$(x-1)^6 = {}^6C_0x^6 - {}^6C_1x^5 + {}^6C_2x^4 - {}^6C_3x^3 + {}^6C_4x^2 - {}^6C_5x + {}^6C_6$$

$$\therefore (x+1)^6 + (x-1)^6 = 2\left[{}^6C_0x^6 + {}^6C_2x^4 + {}^6C_4x^2 + {}^6C_6\right]$$

$$= 2\left[x^6 + 15x^4 + 15x^2 + 1\right]$$

By putting $x = \sqrt{2}$, we obtain

$$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6 = 2\left[\left(\sqrt{2}\right)^6 + 15\left(\sqrt{2}\right)^4 + 15\left(\sqrt{2}\right)^2 + 1\right]$$

$$= 2\left(8 + 15\times4 + 15\times2 + 1\right)$$

$$= 2\left(8 + 60 + 30 + 1\right)$$

$$= 2\left(99\right) = 198$$

Question 13:

Show that $9^{n+1}-8n-9$ is divisible by 64, whenever *n* is a positive integer.

Answer

In order to show that $9^{n+1}-8n-9$ is divisible by 64, it has to be proved that,

$9^{n+1}-8n-9=64k$, where *k* is some natural number

By Binomial Theorem,

$$(1+a)^m = {}^mC_0 + {}^mC_1a + {}^mC_2a^2 + \ldots + {}^mC_ma^m$$

For *a* = 8 and *m* = *n* + 1, we obtain

$$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + \ldots + {}^{n+1}C_{n+1}(8)^{n+1}$$

$$\Rightarrow 9^{n+1} = 1+(n+1)(8)+8^2\left[{}^{n+1}C_2 + {}^{n+1}C_3\times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}\right]$$

$$\Rightarrow 9^{n+1} = 9+8n+64\left[{}^{n+1}C_2 + {}^{n+1}C_3\times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}\right]$$

$$\Rightarrow 9^{n+1}-8n-9 = 64k, \text{ where } k = {}^{n+1}C_2 + {}^{n+1}C_3\times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1} \text{ is a natural number}$$

Thus, $9^{n+1}-8n-9$ is divisible by 64, whenever *n* is a positive integer.

Question 14:

Prove that $\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$.

Answer

By Binomial Theorem,

$$\sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r = (a+b)^n$$

By putting *b* = 3 and *a* = 1 in the above equation, we obtain

$$\sum_{r=0}^{n} {}^nC_r (1)^{n-r} (3)^r = (1+3)^n$$

$$\Rightarrow \sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$$

Hence, proved.